(b) *Washer*: $V = V_1 + V_2$

$V_1 = \int_{a_1}^{b_1} \pi\left([R_1(x)]^2 - [r_1(x)]^2\right) dx$ with $R_1(x) = \sqrt{\frac{x+2}{3}}$ and $r_1(x) = 0$; $a_1 = -2$ and $b_1 = 0$;

$V_2 = \int_{a_2}^{b_2} \pi\left([R_2(x)]^2 - [r_2(x)]^2\right) dx$ with $R_2(x) = \sqrt{\frac{x+2}{3}}$ and $r_2(x) = \sqrt{x}$; $a_2 = 0$ and $b_2 = 1$

$\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_c^d 2\pi y \left(\substack{\text{shell}\\ \text{height}}\right) dy$ where shell height $= y^2 - (3y^2 - 2) = 2 - 2y^2$; $c = 0$ and $d = 1$. Only *one* integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \pi$.

40. (a) *Disk*: $V = V_1 - V_2 - V_3$

$V_i = \int_{c_i}^{d_i} \pi[R_i(y)]^2\, dy$, $i = 1, 2, 3$ with $R_1(y) = 1$ and $c_1 = -1$, $d_1 = 1$; $R_2(y) = \sqrt{y}$ and $c_2 = 0$ and $d_2 = 1$; $R_3(y) = (-y)^{1/4}$ and $c_3 = -1$, $d_3 = 0$ $\Rightarrow$ three integrals are required

(b) *Washer*: $V = V_1 + V_2$

$V_i = \int_{c_i}^{d_i} \pi\left([R_i(y)]^2 - [r_i(y)]^2\right) dy$, $i = 1, 2$ with $R_1(y) = 1$, $r_1(y) = \sqrt{y}$, $c_1 = 0$ and $d_1 = 1$; $R_2(y) = 1$, $r_2(y) = (-y)^{1/4}$, $c_2 = -1$ and $d_2 = 0$ $\Rightarrow$ two integrals are required

(c) *Shell*: $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_a^b 2\pi x \left(\substack{\text{shell}\\ \text{height}}\right) dx$, where shell height $= x^2 - (-x^4) = x^2 + x^4$, $a = 0$ and $b = 1$ $\Rightarrow$ only one integral is required. It is, therefore preferable to use the *shell* method. However, whichever method you use, you will get $V = \frac{5\pi}{6}$.

41. (a) $V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx = \int_{-4}^{4} \pi\left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx = \pi \int_{-4}^{4} \left[25 - x^2 - 9\right] dx = \pi \int_{-4}^{4} (16 - x^2) dx$

$= \pi\left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \pi\left(64 - \frac{64}{3}\right) - \pi\left(-64 + \frac{64}{3}\right) = \frac{256\pi}{3}$

(b) Volume of sphere $= \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3} \Rightarrow$ Volume of portion removed $= \frac{500\pi}{3} - \frac{256\pi}{3} = \frac{244\pi}{3}$

42. $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_1^{\sqrt{1+\pi}} 2\pi x \sin(x^2 - 1)\, dx$; $[u = x^2 - 1 \Rightarrow du = 2x\, dx;\ x = 1 \Rightarrow u = 0,$ $x = \sqrt{1+\pi} \Rightarrow u = \pi] \to \pi \int_0^{\pi} \sin u\, du = -\pi\left[\cos u\right]_0^{\pi} = -\pi(-1 - 1) = 2\pi$

43. $V = \int_a^b 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dx = \int_0^r 2\pi x\left(-\frac{h}{r}x + h\right) dx = 2\pi \int_0^r \left(-\frac{h}{r}x^2 + hx\right) dx = 2\pi\left[-\frac{h}{3r}x^3 + \frac{h}{2}x^2\right]_0^r$

$= 2\pi\left(-\frac{r^2h}{3} + \frac{r^2h}{2}\right) = \frac{1}{3}\pi r^2 h$

44. $V = \int_c^d 2\pi \left(\substack{\text{shell}\\ \text{radius}}\right)\left(\substack{\text{shell}\\ \text{height}}\right) dy = \int_0^r 2\pi y\left[\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2}\right)\right] dy = 4\pi \int_0^r y\sqrt{r^2 - y^2}\, dy$

$[u = r^2 - y^2 \Rightarrow du = -2y\, dy;\ y = 0 \Rightarrow u = r^2,\ y = r \Rightarrow u = 0] \to -2\pi \int_{r^2}^{0} \sqrt{u}\, du = 2\pi \int_0^{r^2} u^{1/2}\, du$

$= \frac{4\pi}{3}\left[u^{3/2}\right]_0^{r^2} = \frac{4\pi}{3} r^3$

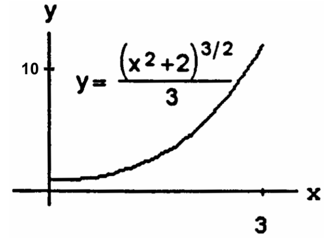
6.3 ARC LENGTHS

1. $\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2}\left(x^2+2\right)^{1/2} \cdot 2x = \sqrt{(x^2+2)} \cdot x$

$\Rightarrow L = \int_0^3 \sqrt{1+(x^2+2)\,x^2}\,dx = \int_0^3 \sqrt{1+2x^2+x^4}\,dx$

$= \int_0^3 \sqrt{(1+x^2)^2}\,dx = \int_0^3 (1+x^2)\,dx = \left[x+\frac{x^3}{3}\right]_0^3$

$= 3 + \frac{27}{3} = 12$

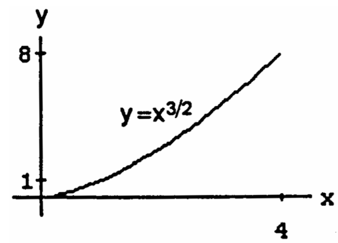


2. $\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$; $\left[u = 1+\frac{9}{4}x\right.$

$\Rightarrow du = \frac{9}{4}\,dx \Rightarrow \frac{4}{9}\,du = dx$; $x = 0 \Rightarrow u = 1$; $x = 4$

$\left.\Rightarrow u = 10\right] \rightarrow L = \int_1^{10} u^{1/2}\left(\frac{4}{9}\,du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$

$= \frac{8}{27}\left(10\sqrt{10}-1\right)$



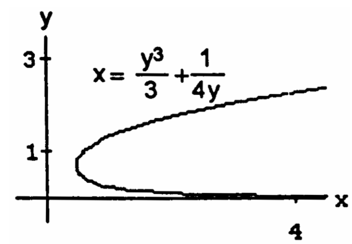
3. $\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$

$\Rightarrow L = \int_1^3 \sqrt{1+y^4-\frac{1}{2}+\frac{1}{16y^4}}\,dy$

$= \int_1^3 \sqrt{y^4+\frac{1}{2}+\frac{1}{16y^4}}\,dy$

$= \int_1^3 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy = \int_1^3 \left(y^2+\frac{1}{4y^2}\right)dy$

$= \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3 = \left(\frac{27}{3}-\frac{1}{12}\right) - \left(\frac{1}{3}-\frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = 9 + \frac{(-1-4+3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$



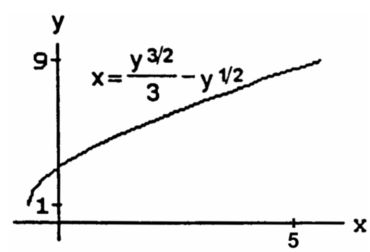
4. $\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y-2+\frac{1}{y}\right)$

$\Rightarrow L = \int_1^9 \sqrt{1+\frac{1}{4}\left(y-2+\frac{1}{y}\right)}\,dy$

$= \int_1^9 \sqrt{\frac{1}{4}\left(y+2+\frac{1}{y}\right)}\,dy = \int_1^9 \frac{1}{2}\sqrt{\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2}\,dy$

$= \frac{1}{2}\int_1^9 \left(y^{1/2}+y^{-1/2}\right)dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9$

$= \left[\frac{y^{3/2}}{3}+y^{1/2}\right]_1^9 = \left(\frac{3^3}{3}+3\right) - \left(\frac{1}{3}+1\right) = 11 - \frac{1}{3} = \frac{32}{3}$



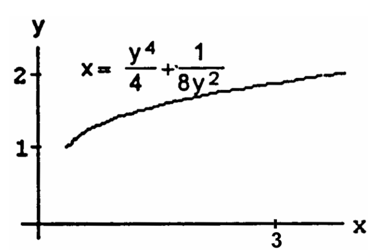
5. $\frac{dx}{dy} = y^3 - \frac{1}{4y^3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^6 - \frac{1}{2} + \frac{1}{16y^6}$

$\Rightarrow L = \int_1^2 \sqrt{1+y^6-\frac{1}{2}+\frac{1}{16y^6}}\,dy$

$= \int_1^2 \sqrt{y^6+\frac{1}{2}+\frac{1}{16y^6}}\,dy = \int_1^2 \sqrt{\left(y^3+\frac{y^{-3}}{4}\right)^2}\,dy$

$= \int_1^2 \left(y^3+\frac{y^{-3}}{4}\right)dy = \left[\frac{y^4}{4}-\frac{y^{-2}}{8}\right]_1^2$

$= \left(\frac{16}{4}-\frac{1}{(16)(2)}\right) - \left(\frac{1}{4}-\frac{1}{8}\right) = 4 - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{128-1-8+4}{32} = \frac{123}{32}$



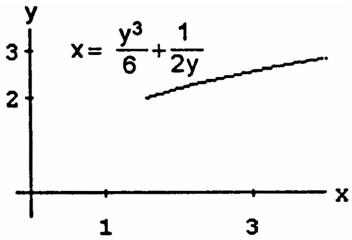
6. $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)$

$\Rightarrow L = \int_2^3 \sqrt{1 + \frac{1}{4}\left(y^4 - 2 + y^{-4}\right)}\, dy$

$= \int_2^3 \sqrt{\frac{1}{4}\left(y^4 + 2 + y^{-4}\right)}\, dy$

$= \frac{1}{2}\int_2^3 \sqrt{\left(y^2 + y^{-2}\right)^2}\, dy = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right) dy$

$= \frac{1}{2}\left[\frac{y^3}{3} - y^{-1}\right]_2^3 = \frac{1}{2}\left[\left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right] = \frac{1}{2}\left(\frac{26}{3} - \frac{8}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(6 + \frac{1}{2}\right) = \frac{13}{4}$



7. $\frac{dy}{dx} = x^{1/3} - \frac{1}{4}x^{-1/3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}$

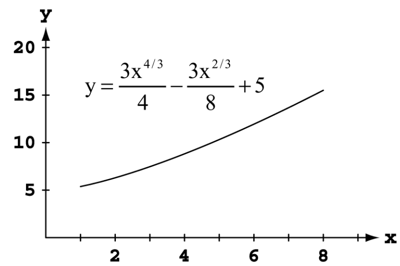
$\Rightarrow L = \int_1^8 \sqrt{1 + x^{2/3} - \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{x^{2/3} + \frac{1}{2} + \frac{x^{-2/3}}{16}}\, dx$

$= \int_1^8 \sqrt{\left(x^{1/3} + \frac{1}{4}x^{-1/3}\right)^2}\, dx = \int_1^8 \left(x^{1/3} + \frac{1}{4}x^{-1/3}\right) dx$

$= \left[\frac{3}{4}x^{4/3} + \frac{3}{8}x^{2/3}\right]_1^8 = \frac{3}{8}\left[2x^{4/3} + x^{2/3}\right]_1^8$

$= \frac{3}{8}\left[\left(2 \cdot 2^4 + 2^2\right) - (2+1)\right] = \frac{3}{8}(32 + 4 - 3) = \frac{99}{8}$



8. $\frac{dy}{dx} = x^2 + 2x + 1 - \frac{4}{(4x+4)^2} = x^2 + 2x + 1 - \frac{1}{4}\,\frac{1}{(1+x)^2}$

$= (1+x)^2 - \frac{1}{4}\,\frac{1}{(1+x)^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+x)^4 - \frac{1}{2} + \frac{1}{16(1+x)^4}$

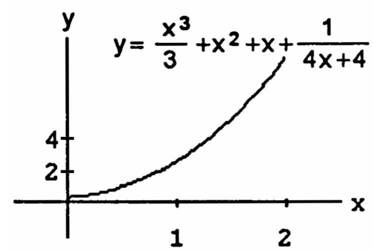
$\Rightarrow L = \int_0^2 \sqrt{1 + (1+x)^4 - \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{(1+x)^4 + \frac{1}{2} + \frac{(1+x)^{-4}}{16}}\, dx$

$= \int_0^2 \sqrt{\left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right]^2}\, dx$

$= \int_0^2 \left[(1+x)^2 + \frac{(1+x)^{-2}}{4}\right] dx$; $[u = 1 + x \Rightarrow du = dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3]$

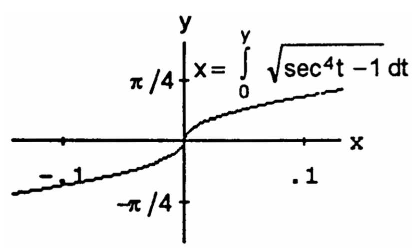
$\to L = \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du = \left[\frac{u^3}{3} - \frac{1}{4}u^{-1}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}$



9. $\frac{dx}{dy} = \sqrt{\sec^4 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - 1$

$\Rightarrow L = \int_{-\pi/4}^{\pi/4} \sqrt{1 + \left(\sec^4 y - 1\right)}\, dy = \int_{-\pi/4}^{\pi/4} \sec^2 y\, dy$

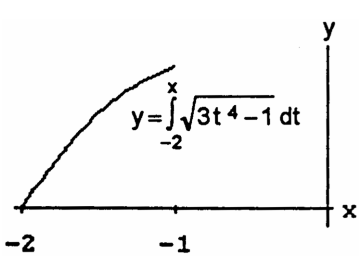
$= [\tan y]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$



10. $\frac{dy}{dx} = \sqrt{3x^4 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 3x^4 - 1$

$\Rightarrow L = \int_{-2}^{-1} \sqrt{1 + \left(3x^4 - 1\right)}\, dx = \int_{-2}^{-1} \sqrt{3}\, x^2\, dx$

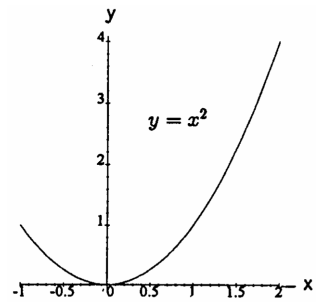
$= \sqrt{3}\left[\frac{x^3}{3}\right]_{-2}^{-1} = \frac{\sqrt{3}}{3}\left[-1 - (-2)^3\right] = \frac{\sqrt{3}}{3}(-1 + 8) = \frac{7\sqrt{3}}{3}$



11. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$

$$\Rightarrow L = \int_{-1}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{-1}^{2} \sqrt{1 + 4x^2}\, dx$$

(b)

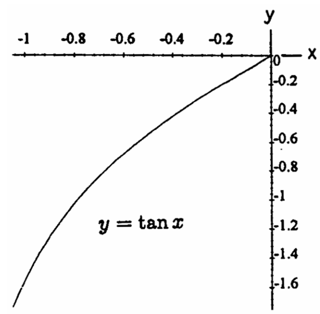


(c) $L \approx 6.13$

12. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$$\Rightarrow L = \int_{-\pi/3}^{0} \sqrt{1 + \sec^4 x}\, dx$$

(b)

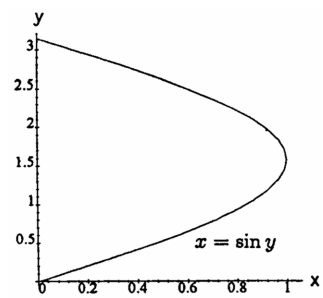


(c) $L \approx 2.06$

13. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$

$$\Rightarrow L = \int_{0}^{\pi} \sqrt{1 + \cos^2 y}\, dy$$

(b)

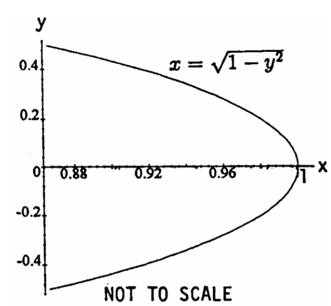


(c) $L \approx 3.82$

14. (a) $\frac{dx}{dy} = -\frac{y}{\sqrt{1-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{1-y^2}$

$$\Rightarrow L = \int_{-1/2}^{1/2} \sqrt{1 + \frac{y^2}{(1-y^2)}}\, dy = \int_{-1/2}^{1/2} \sqrt{\frac{1}{1-y^2}}\, dy$$
$$= \int_{-1/2}^{1/2} (1-y^2)^{-1/2}\, dy$$

(b)

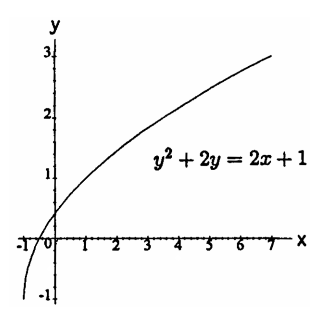


(c) $L \approx 1.05$

15. (a) $2y + 2 = 2\,\frac{dx}{dy} \Rightarrow \left(\frac{dx}{dy}\right)^2 = (y+1)^2$

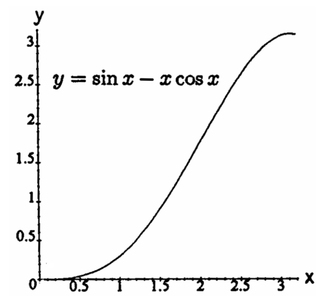
$$\Rightarrow L = \int_{-1}^{3} \sqrt{1 + (y+1)^2}\, dy$$

(b)



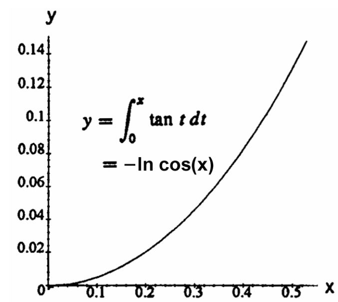
(c) $L \approx 9.29$

16. (a) $\frac{dy}{dx} = \cos x - \cos x + x \sin x \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 \sin^2 x$
$\Rightarrow L = \int_0^{\pi} \sqrt{1 + x^2 \sin^2 x}\, dx$
(b)



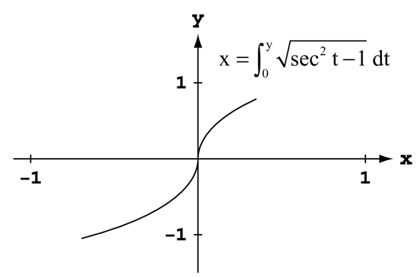
(c) $L \approx 4.70$

17. (a) $\frac{dy}{dx} = \tan x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x$
$\Rightarrow L = \int_0^{\pi/6} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/6} \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx$
$= \int_0^{\pi/6} \frac{dx}{\cos x} = \int_0^{\pi/6} \sec x\, dx$
(b)



(c) $L \approx 0.55$

18. (a) $\frac{dx}{dy} = \sqrt{\sec^2 y - 1} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y - 1$
$\Rightarrow L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + (\sec^2 y - 1)}\, dy$
$= \int_{-\pi/3}^{\pi/4} |\sec y|\, dy = \int_{-\pi/3}^{\pi/4} \sec y\, dy$
(b)



(c) $L \approx 2.20$

19. (a) $\left(\frac{dy}{dx}\right)^2$ corresponds to $\frac{1}{4x}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{2\sqrt{x}}$. Then $y = \sqrt{x} + C$ and since $(1, 1)$ lies on the curve, $C = 0$. So $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.
(b) Only one. We know the derivative of the function and the value of the function at one value of x.

20. (a) $\left(\frac{dx}{dy}\right)^2$ corresponds to $\frac{1}{y^4}$ here, so take $\frac{dy}{dx}$ as $\frac{1}{y^2}$. Then $x = -\frac{1}{y} + C$ and, since $(0, 1)$ lies on the curve, $C = 1$. So $y = \frac{1}{1-x}$.
(b) Only one. We know the derivative of the function and the value of the function at one value of x.

21. $y = \int_0^x \sqrt{\cos 2t}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow L = \int_0^{\pi/4} \sqrt{1 + \left[\sqrt{\cos 2x}\right]^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \cos 2x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx$
$= \int_0^{\pi/4} \sqrt{2} \cos x\, dx = \sqrt{2}[\sin x]_0^{\pi/4} = \sqrt{2}\sin\left(\frac{\pi}{4}\right) - \sqrt{2}\sin(0) = 1$

22. $y = \left(1 - x^{2/3}\right)^{3/2}, \frac{\sqrt{2}}{4} \le x \le 1 \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\left(1 - x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow L = \int_{\sqrt{2}/4}^{1} \sqrt{1 + \left[-\frac{\left(1 - x^{2/3}\right)^{1/2}}{x^{1/3}}\right]^2}\, dx$
$= \int_{\sqrt{2}/4}^{1} \sqrt{1 + \frac{1 - x^{2/3}}{x^{2/3}}}\, dx = \int_{\sqrt{2}/4}^{1} \sqrt{1 + \frac{1}{x^{2/3}} - 1}\, dx = \int_{\sqrt{2}/4}^{1} \sqrt{\frac{1}{x^{2/3}}}\, dx = \int_{\sqrt{2}/4}^{1} \frac{1}{x^{1/3}}\, dx = \int_{\sqrt{2}/4}^{1} x^{-1/3}\, dx = \frac{3}{2}\left[x^{2/3}\right]_{\sqrt{2}/4}^{1}$
$= \frac{3}{2}(1)^{2/3} - \frac{3}{2}\left(\frac{\sqrt{2}}{4}\right)^{2/3} = \frac{3}{2} - \frac{3}{2}\left(\frac{1}{2}\right) = \frac{3}{4} \Rightarrow \text{total length} = 8\left(\frac{3}{4}\right) = 6$

23. $y = 3 - 2x,\ 0 \le x \le 2 \Rightarrow \frac{dy}{dx} = -2 \Rightarrow\ L = \int_0^2 \sqrt{1+(-2)^2}\,dx = \int_0^2 \sqrt{5}\,dx = \left[\sqrt{5}\,x\right]_0^2 = 2\sqrt{5}.$

$d = \sqrt{(2-0)^2 + (3-(-1))^2} = 2\sqrt{5}$

24. Consider the circle $x^2 + y^2 = r^2$, we will find the length of the portion in the first quadrant, and multiply our result by 4.

$y = \sqrt{r^2 - x^2},\ 0 \le x \le r \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}} \Rightarrow\ L = 4\int_0^r \sqrt{1 + \left[\frac{-x}{\sqrt{r^2-x^2}}\right]^2}\,dx = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2-x^2}}\,dx = 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\,dx$

$= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx = 4r\int_0^r \frac{dx}{\sqrt{r^2-x^2}}$

25. $9x^2 = y(y-3)^2 \Rightarrow \frac{d}{dy}\left[9x^2\right] = \frac{d}{dy}\left[y(y-3)^2\right] \Rightarrow 18x\frac{dx}{dy} = 2y(y-3) + (y-3)^2 = 3(y-3)(y-1) \Rightarrow \frac{dx}{dy} = \frac{(y-3)(y-1)}{6x}$

$\Rightarrow dx = \frac{(y-3)(y-1)}{6x}dy;\ ds^2 = dx^2 + dy^2 = \left[\frac{(y-3)(y-1)}{6x}dy\right]^2 + dy^2 = \frac{(y-3)^2(y-1)^2}{36x^2}dy^2 + dy^2 = \frac{(y-3)^2(y-1)^2}{4y(y-3)^2}dy^2 + dy^2$

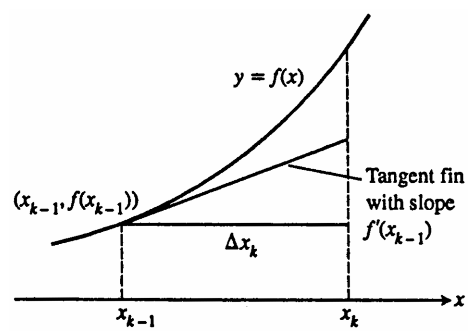
$= \left[\frac{(y-1)^2}{4y} + 1\right]dy^2 = \frac{y^2-2y+1+4y}{4y}dy^2 = \frac{(y+1)^2}{4y}dy^2$

26. $4x^2 - y^2 = 64 \Rightarrow \frac{d}{dx}\left[4x^2 - y^2\right] = \frac{d}{dx}\left[64\right] \Rightarrow 8x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{y} \Rightarrow dy = \frac{4x}{y}dx;\ ds^2 = dx^2 + dy^2$

$= dx^2 + \left[\frac{4x}{y}dx\right]^2 = dx^2 + \frac{16x^2}{y^2}dx^2 = \left(1 + \frac{16x^2}{y^2}\right)dx^2 = \frac{y^2+16x^2}{y^2}dx^2 = \frac{4x^2-64+16x^2}{y^2}dx^2 = \frac{20x^2-64}{y^2}dx^2 = \frac{4}{y^2}(5x^2-16)dx^2$

27. $\sqrt{2}\,x = \int_0^x \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\,dt,\ x \ge 0 \Rightarrow \sqrt{2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{dy}{dx} = \pm 1 \Rightarrow y = f(x) = \pm x + C$ where C is any real number.

28. (a) From the accompanying figure and definition of the differential (change along the tangent line) we see that $dy = f'(x_{k-1})\,\triangle x_k\ \Rightarrow$ length of kth tangent fin is $\sqrt{(\triangle x_k)^2 + (dy)^2} = \sqrt{(\triangle x_k)^2 + [f'(x_{k-1})\,\triangle x_k]^2}$.



(b) Length of curve $= \lim_{n\to\infty} \sum_{k=1}^{n}$ (length of kth tangent fin) $= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(\triangle x_k)^2 + [f'(x_{k-1})\,\triangle x_k]^2}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(x_{k-1})]^2}\,\triangle x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

29. $x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-x^2};\ P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \Rightarrow L \approx \sum_{k=1}^{4} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\left(\frac{1}{4} - 0\right)^2 + \left(\frac{\sqrt{15}}{4} - 1\right)^2}$

$+ \sqrt{\left(\frac{1}{2} - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{15}}{4}\right)^2} + \sqrt{\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{4} - \frac{\sqrt{3}}{2}\right)^2} + \sqrt{\left(1 - \frac{3}{4}\right)^2 + \left(0 - \frac{\sqrt{7}}{4}\right)^2} \approx 1.55225$

30. Let (x_1, y_1) and (x_2, y_2), with $x_2 > x_1$, lie on $y = mx + b$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$, then $\frac{dy}{dx} = m \Rightarrow L = \int_{x_1}^{x_2} \sqrt{1+m^2}\,dx$

$= \sqrt{1+m^2}\,[x]_{x_1}^{x_2} = \sqrt{1+m^2}(x_2 - x_1) = \sqrt{1 + \left(\frac{y_2-y_1}{x_2-x_1}\right)^2}(x_2 - x_1) = \sqrt{\frac{(x_2-x_1)^2 + (y_2-y_1)^2}{(x_2-x_1)^2}}(x_2 - x_1)$

$\frac{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}{(x_2-x_1)}(x_2 - x_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$

31. $y = 2x^{3/2} \Rightarrow \frac{dy}{dx} = 3x^{1/2}$; $L(x) = \int_0^x \sqrt{1 + \left[3t^{1/2}\right]^2}\,dt = \int_0^x \sqrt{1+9t}\,dt$; $[u = 1 + 9t \Rightarrow du = 9dt, t = 0 \Rightarrow u = 1,$
$t = x \Rightarrow u = 1 + 9x] \rightarrow \frac{1}{9}\int_1^{1+9x} \sqrt{u}\,du = \frac{2}{27}\left[u^{3/2}\right]_1^{1+9x} = \frac{2}{27}(1+9x)^{3/2} - \frac{2}{27}$; $L(1) = \frac{2}{27}(10)^{3/2} - \frac{2}{27} = \frac{2\left(10\sqrt{10}-1\right)}{27}$

32. $y = \frac{x^3}{3} + x^2 + x + \frac{1}{4x+4} \Rightarrow \frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4(x+1)^2} = (x+1)^2 - \frac{1}{4(x+1)^2}$;

$L(x) = \int_0^x \sqrt{1 + \left[(t+1)^2 - \frac{1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1 + \left[\frac{4(t+1)^4 - 1}{4(t+1)^2}\right]^2}\,dt = \int_0^x \sqrt{1 + \frac{\left[4(t+1)^4 - 1\right]^2}{16(t+1)^4}}\,dt$

$= \int_0^x \sqrt{\frac{16(t+1)^4 + 16(t+1)^8 - 8(t+1)^4 + 1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\frac{16(t+1)^8 + 8(t+1)^4 + 1}{16(t+1)^4}}\,dt = \int_0^x \sqrt{\frac{\left[4(t+1)^4 + 1\right]^2}{16(t+1)^4}}\,dt$

$= \int_0^x \frac{4(t+1)^4 + 1}{4(t+1)^2}\,dt = \int_0^x \left[(t+1)^2 + \frac{1}{4(t+1)^2}\right]dt$; $[u = t + 1 \Rightarrow du = dt, t = 0 \Rightarrow u = 1, t = x \Rightarrow u = x + 1]$

$\rightarrow \int_1^{x+1} \left[u^2 + \frac{1}{4}u^{-2}\right]du = \left[\frac{1}{3}u^3 - \frac{1}{4}u^{-1}\right]_1^{x+1} = \left(\frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{3}(x+1)^3 - \frac{1}{4(x+1)} - \frac{1}{12}$;

$L(1) = \frac{8}{3} - \frac{1}{8} - \frac{1}{12} = \frac{59}{24}$

33-38. Example CAS commands:

Maple:

```
with( plots );
with( Student[Calculus1] );
with( student );
f := x -> sqrt(1-x^2);a := -1;
b := 1;
N := [2, 4, 8 ];
for n in N do
  xx := [seq( a+i*(b-a)/n, i=0..n )];
  pts := [seq([x,f(x)],x=xx)];
  L := simplify(add( distance(pts[i+1],pts[i]), i=1..n ));                    # (b)
  T := sprintf("#33(a) (Section 6.3)\nn=%3d  L=%8.5f\n", n, L );
  P[n] := plot( [f(x),pts], x=a..b, title=T ):                                   # (a)
end do:
display( [seq(P[n],n=N)], insequence=true, scaling=constrained );
L := ArcLength( f(x), x=a..b, output=integral ):
L = evalf( L );                                                                   # (c)
```

33-38. Example CAS commands:

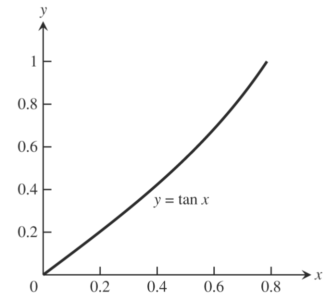
Mathematica: (assigned function and values for a, b, and n may vary)

```
Clear[x, f]
{a, b} = {−1, 1}; f[x_] = Sqrt[1 − x^2]
p1 = Plot[f[x], {x, a, b}]
n = 8;
pts = Table[{xn, f[xn]}, {xn, a, b, (b − a)/n}]/ / N
Show[{p1,Graphics[{Line[pts]}]}]
Sum[ Sqrt[ (pts[[i + 1, 1]] − pts[[i, 1]])^2 + (pts[[i + 1, 2]] − pts[[i, 2]])^2], {i, 1, n}]
NIntegrate[ Sqrt[ 1 + f'[x]^2],{x, a, b}]
```

6.4 AREAS OF SURFACES OF REVOLUTION

1. (a) $\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$
$\Rightarrow S = 2\pi \int_0^{\pi/4} (\tan x) \sqrt{1 + \sec^4 x}\, dx$

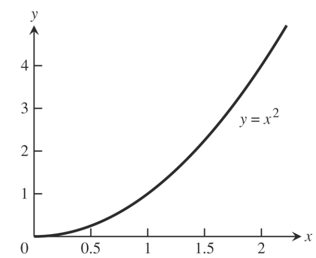
(b)



(c) $S \approx 3.84$

2. (a) $\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$
$\Rightarrow S = 2\pi \int_0^2 x^2 \sqrt{1 + 4x^2}\, dx$

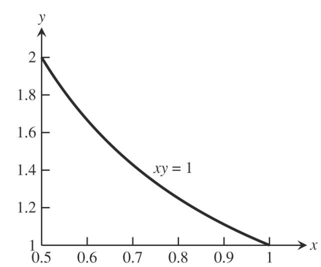
(b)



(c) $S \approx 53.23$

3. (a) $xy = 1 \Rightarrow x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^4}$
$\Rightarrow S = 2\pi \int_1^2 \frac{1}{y} \sqrt{1 + y^{-4}}\, dy$

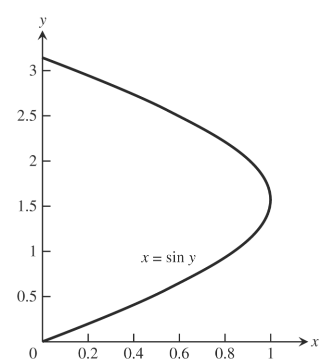
(b)



(c) $S \approx 5.02$

4. (a) $\frac{dx}{dy} = \cos y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \cos^2 y$
$\Rightarrow S = 2\pi \int_0^{\pi} (\sin y) \sqrt{1 + \cos^2 y}\, dy$

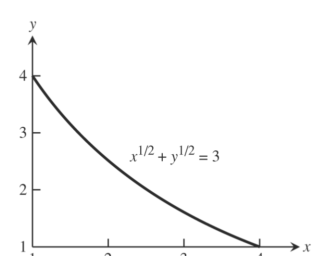
(b)



(c) $S \approx 14.42$

5. (a) $x^{1/2} + y^{1/2} = 3 \Rightarrow y = \left(3 - x^{1/2}\right)^2$
$\Rightarrow \frac{dy}{dx} = 2\left(3 - x^{1/2}\right)\left(-\frac{1}{2}x^{-1/2}\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(1 - 3x^{-1/2}\right)^2$
$\Rightarrow S = 2\pi \int_1^4 \left(3 - x^{1/2}\right)^2 \sqrt{1 + \left(1 - 3x^{-1/2}\right)^2}\, dx$

(b)

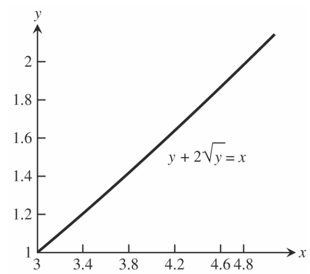


(c) $S \approx 63.37$

6. (a) $\frac{dx}{dy} = 1 + y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(1 + y^{-1/2}\right)^2$

$\Rightarrow S = 2\pi \int_1^2 \left(y + 2\sqrt{y}\right) \sqrt{1 + \left(1 + y^{-1/2}\right)^2}\, dx$

(b)



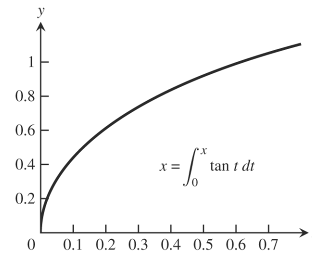
(c) $S \approx 51.33$

7. (a) $\frac{dx}{dy} = \tan y \Rightarrow \left(\frac{dx}{dy}\right)^2 = \tan^2 y$

$\Rightarrow S = 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t\, dt\right) \sqrt{1 + \tan^2 y}\, dy$

$= 2\pi \int_0^{\pi/3} \left(\int_0^y \tan t\, dt\right) \sec y\, dy$

(b)



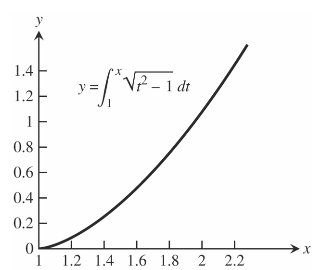
(c) $S \approx 2.08$

8. (a) $\frac{dy}{dx} = \sqrt{x^2 - 1} \Rightarrow \left(\frac{dy}{dx}\right)^2 = x^2 - 1$

$\Rightarrow S = 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1}\, dt\right) \sqrt{1 + (x^2 - 1)}\, dx$

$= 2\pi \int_1^{\sqrt{5}} \left(\int_1^x \sqrt{t^2 - 1}\, dt\right) x\, dx$

(b)



(c) $S \approx 8.55$

9. $y = \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow S = \int_0^4 2\pi \left(\frac{x}{2}\right) \sqrt{1 + \frac{1}{4}}\, dx = \frac{\pi\sqrt{5}}{2} \int_0^4 x\, dx$

$= \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2}\right]_0^4 = 4\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(2)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(4\pi)\left(2\sqrt{5}\right) = 4\pi\sqrt{5}$ in agreement with the integral value

10. $y = \frac{x}{2} \Rightarrow x = 2y \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi \cdot 2y\sqrt{1 + 2^2}\, dy = 4\pi\sqrt{5} \int_0^2 y\, dy = 2\pi\sqrt{5}\left[y^2\right]_0^2$

$= 2\pi\sqrt{5} \cdot 4 = 8\pi\sqrt{5}$; Geometry formula: base circumference $= 2\pi(4)$, slant height $= \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$\Rightarrow$ Lateral surface area $= \frac{1}{2}(8\pi)\left(2\sqrt{5}\right) = 8\pi\sqrt{5}$ in agreement with the integral value

11. $\frac{dy}{dx} = \frac{1}{2}$; $S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^3 2\pi \frac{(x+1)}{2} \sqrt{1 + \left(\frac{1}{2}\right)^2}\, dx = \frac{\pi\sqrt{5}}{2} \int_1^3 (x + 1)\, dx = \frac{\pi\sqrt{5}}{2} \left[\frac{x^2}{2} + x\right]_1^3$

$= \frac{\pi\sqrt{5}}{2} \left[\left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} + 1\right)\right] = \frac{\pi\sqrt{5}}{2}(4 + 2) = 3\pi\sqrt{5}$; Geometry formula: $r_1 = \frac{1}{2} + \frac{1}{2} = 1$, $r_2 = \frac{3}{2} + \frac{1}{2} = 2$,

slant height $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5} \Rightarrow$ Frustum surface area $= \pi(r_1 + r_2) \times$ slant height $= \pi(1 + 2)\sqrt{5}$

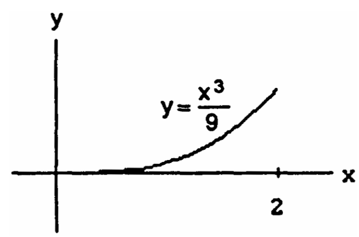
$= 3\pi\sqrt{5}$ in agreement with the integral value

12. $y = \frac{x}{2} + \frac{1}{2} \Rightarrow x = 2y - 1 \Rightarrow \frac{dx}{dy} = 2$; $S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 2\pi(2y - 1)\sqrt{1 + 4}\, dy = 2\pi\sqrt{5} \int_1^2 (2y - 1)\, dy$

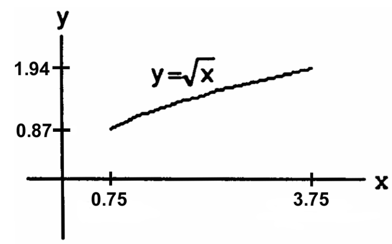
$= 2\pi\sqrt{5}\left[y^2 - y\right]_1^2 = 2\pi\sqrt{5}\left[(4 - 2) - (1 - 1)\right] = 4\pi\sqrt{5}$; Geometry formula: $r_1 = 1$, $r_2 = 3$,

slant height $= \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5} \Rightarrow$ Frustum surface area $= \pi(1+3)\sqrt{5} = 4\pi\sqrt{5}$ in agreement with the integral value

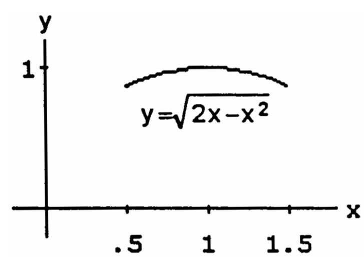
13. $\frac{dy}{dx} = \frac{x^2}{3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{9} \Rightarrow S = \int_0^2 \frac{2\pi x^3}{9}\sqrt{1+\frac{x^4}{9}}\,dx;$
$\left[u = 1 + \frac{x^4}{9} \Rightarrow du = \frac{4}{9}x^3\,dx \Rightarrow \frac{1}{4}\,du = \frac{x^3}{9}\,dx;\right.$
$\left. x = 0 \Rightarrow u = 1, x = 2 \Rightarrow u = \frac{25}{9}\right]$
$\rightarrow S = 2\pi\int_1^{25/9} u^{1/2} \cdot \frac{1}{4}\,du = \frac{\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{25/9}$
$= \frac{\pi}{3}\left(\frac{125}{27} - 1\right) = \frac{\pi}{3}\left(\frac{125-27}{27}\right) = \frac{98\pi}{81}$



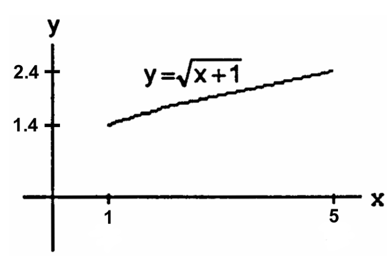
14. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4x}$
$\Rightarrow S = \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$
$= 2\pi\int_{3/4}^{15/4}\sqrt{x+\frac{1}{4}}\,dx = 2\pi\left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}\right]_{3/4}^{15/4}$
$= \frac{4\pi}{3}\left[\left(\frac{15}{4}+\frac{1}{4}\right)^{3/2} - \left(\frac{3}{4}+\frac{1}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left[\left(\frac{4}{2}\right)^3 - 1\right]$
$= \frac{4\pi}{3}(8-1) = \frac{28\pi}{3}$



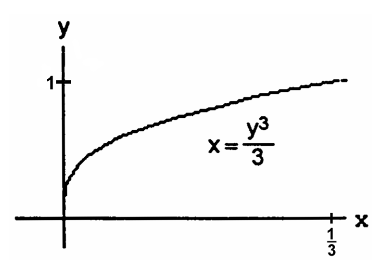
15. $\frac{dy}{dx} = \frac{1}{2}\frac{(2-2x)}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x-x^2}$
$\Rightarrow S = \int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\frac{(1-x)^2}{2x-x^2}}\,dx$
$= 2\pi\int_{0.5}^{1.5}\sqrt{2x-x^2}\,\frac{\sqrt{2x-x^2+1-2x+x^2}}{\sqrt{2x-x^2}}\,dx$
$= 2\pi\int_{0.5}^{1.5} dx = 2\pi[x]_{0.5}^{1.5} = 2\pi$



16. $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4(x+1)}$
$\Rightarrow S = \int_1^5 2\pi\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx$
$= 2\pi\int_1^5\sqrt{(x+1)+\frac{1}{4}}\,dx = 2\pi\int_1^5\sqrt{x+\frac{5}{4}}\,dx$
$= 2\pi\left[\frac{2}{3}\left(x+\frac{5}{4}\right)^{3/2}\right]_1^5 = \frac{4\pi}{3}\left[\left(5+\frac{5}{4}\right)^{3/2} - \left(1+\frac{5}{4}\right)^{3/2}\right]$
$= \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{3/2} - \left(\frac{9}{4}\right)^{3/2}\right] = \frac{4\pi}{3}\left(\frac{5^3}{2^3} - \frac{3^3}{2^3}\right)$
$= \frac{\pi}{6}(125-27) = \frac{98\pi}{6} = \frac{49\pi}{3}$



17. $\frac{dx}{dy} = y^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 \Rightarrow S = \int_0^1 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy;$
$\left[u = 1 + y^4 \Rightarrow du = 4y^3\,dy \Rightarrow \frac{1}{4}\,du = y^3\,dy;\ y = 0\right.$
$\left.\Rightarrow u = 1, y = 1 \Rightarrow u = 2\right] \rightarrow S = \int_1^2 2\pi\left(\frac{1}{3}\right)u^{1/2}\left(\frac{1}{4}\,du\right)$
$= \frac{\pi}{6}\int_1^2 u^{1/2}\,du = \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{\pi}{9}\left(\sqrt{8}-1\right)$

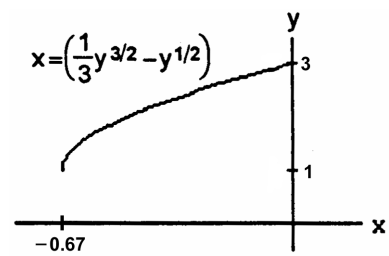


18. $x = \left(\frac{1}{3}\,y^{3/2} - y^{1/2}\right) \le 0$, when $1 \le y \le 3$. To get positive area, we take $x = -\left(\frac{1}{3}\,y^{3/2} - y^{1/2}\right)$

$\Rightarrow\ \frac{dx}{dy} = -\frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)\ \Rightarrow\ \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + y^{-1}\right)$

$\Rightarrow\ S = -\int_1^3 2\pi\left(\frac{1}{3}\,y^{3/2} - y^{1/2}\right)\sqrt{1 + \frac{1}{4}\left(y - 2 + y^{-1}\right)}\,dy$

$= -2\pi\int_1^3\left(\frac{1}{3}\,y^{3/2} - y^{1/2}\right)\sqrt{\frac{1}{4}\left(y + 2 + y^{-1}\right)}\,dy$



$= -2\pi\int_1^3\left(\frac{1}{3}\,y^{3/2} - y^{1/2}\right)\frac{\sqrt{\left(y^{1/2}+y^{-1/2}\right)^2}}{2}\,dy = -\pi\int_1^3 y^{1/2}\left(\frac{1}{3}\,y - 1\right)\left(y^{1/2} + \frac{1}{y^{1/2}}\right)dy = -\pi\int_1^3\left(\frac{1}{3}\,y - 1\right)(y+1)\,dy$

$= -\pi\int_1^3\left(\frac{1}{3}\,y^2 - \frac{2}{3}\,y - 1\right)dy = -\pi\left[\frac{y^3}{9} - \frac{y^2}{3} - y\right]_1^3 = -\pi\left[\left(\frac{27}{9} - \frac{9}{3} - 3\right) - \left(\frac{1}{9} - \frac{1}{3} - 1\right)\right] = -\pi\left(-3 - \frac{1}{9} + \frac{1}{3} + 1\right)$

$= -\frac{\pi}{9}(-18 - 1 + 3) = \frac{16\pi}{9}$

19. $\frac{dx}{dy} = \frac{-1}{\sqrt{4-y}}\ \Rightarrow\ \left(\frac{dx}{dy}\right)^2 = \frac{1}{4-y}\ \Rightarrow\ S = \int_0^{15/4} 2\pi\cdot 2\sqrt{4-y}\,\sqrt{1 + \frac{1}{4-y}}\,dy = 4\pi\int_0^{15/4}\sqrt{(4-y)+1}\,dy$

$= 4\pi\int_0^{15/4}\sqrt{5-y}\,dy = -4\pi\left[\frac{2}{3}(5-y)^{3/2}\right]_0^{15/4} = -\frac{8\pi}{3}\left[\left(5 - \frac{15}{4}\right)^{3/2} - 5^{3/2}\right] = -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right]$

$= \frac{8\pi}{3}\left(5\sqrt{5} - \frac{5\sqrt{5}}{8}\right) = \frac{8\pi}{3}\left(\frac{40\sqrt{5}-5\sqrt{5}}{8}\right) = \frac{35\pi\sqrt{5}}{3}$

20. $\frac{dx}{dy} = \frac{1}{\sqrt{2y-1}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{2y-1}\ \Rightarrow\ S = \int_{5/8}^1 2\pi\sqrt{2y-1}\,\sqrt{1 + \frac{1}{2y-1}}\,dy = 2\pi\int_{5/8}^1\sqrt{(2y-1)+1}\,dy = 2\pi\int_{5/8}^1\sqrt{2}\,y^{1/2}\,dy$

$= 2\pi\sqrt{2}\left[\frac{2}{3}\,y^{3/2}\right]_{5/8}^1 = \frac{4\pi\sqrt{2}}{3}\left[1^{3/2} - \left(\frac{5}{8}\right)^{3/2}\right] = \frac{4\pi\sqrt{2}}{3}\left(1 - \frac{5\sqrt{5}}{8\sqrt{8}}\right) = \frac{4\pi\sqrt{2}}{3}\left(\frac{8\cdot 2\sqrt{2} - 5\sqrt{5}}{8\cdot 2\sqrt{2}}\right) = \frac{\pi}{12}\left(16\sqrt{2} - 5\sqrt{5}\right)$

21. $S = 2\pi\int_{1/2}^1\sqrt{2y-1}\,\sqrt{1 + \left(\frac{1}{\sqrt{2y-1}}\right)^2}\,dy = 2\pi\int_{1/2}^1\sqrt{2y-1}\,\sqrt{1 + \frac{1}{2y-1}}\,dy = 2\pi\int_{1/2}^1\sqrt{2y-1}\,\sqrt{\frac{2y}{2y-1}}\,dy$

$= 2\pi\int_{1/2}^1\sqrt{2y}\,dy = 2\sqrt{2}\,\pi\int_{1/2}^1\sqrt{y}\,dy = 2\sqrt{2}\,\pi\left[\frac{2}{3}\,y^{3/2}\right]_{1/2}^1 = 2\sqrt{2}\,\pi\left[\left(\frac{2}{3}\sqrt{1^3}\right) - \left(\frac{2}{3}\sqrt{\left(\frac{1}{2}\right)^3}\right)\right] = 2\sqrt{2}\,\pi\left(\frac{2}{3} - \frac{1}{3\sqrt{2}}\right)$

$= 2\sqrt{2}\,\pi\left(\frac{2\sqrt{2}-1}{3\sqrt{2}}\right) = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right)$

22. $y = \frac{1}{3}\left(x^2+2\right)^{3/2}\ \Rightarrow\ dy = x\sqrt{x^2+2}\,dx\ \Rightarrow\ ds = \sqrt{1 + (2x^2 + x^4)}\,dx\ \Rightarrow\ S = 2\pi\int_0^{\sqrt{2}} x\sqrt{1 + 2x^2 + x^4}\,dx$

$= 2\pi\int_0^{\sqrt{2}} x\sqrt{\left(x^2+1\right)^2}\,dx = 2\pi\int_0^{\sqrt{2}} x\left(x^2+1\right)dx = 2\pi\int_0^{\sqrt{2}}\left(x^3 + x\right)dx = 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^{\sqrt{2}} = 2\pi\left(\frac{4}{4} + \frac{2}{2}\right) = 4\pi$

23. $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(y^3 - \frac{1}{4y^3}\right)^2 + 1}\,dy = \sqrt{\left(y^6 - \frac{1}{2} + \frac{1}{16y^6}\right) + 1}\,dy = \sqrt{\left(y^6 + \frac{1}{2} + \frac{1}{16y^6}\right)}\,dy$

$= \sqrt{\left(y^3 + \frac{1}{4y^3}\right)^2}\,dy = \left(y^3 + \frac{1}{4y^3}\right)dy$; $S = \int_1^2 2\pi y\,ds = 2\pi\int_1^2 y\left(y^3 + \frac{1}{4y^3}\right)dy = 2\pi\int_1^2\left(y^4 + \frac{1}{4}\,y^{-2}\right)dy$

$= 2\pi\left[\frac{y^5}{5} - \frac{1}{4}\,y^{-1}\right]_1^2 = 2\pi\left[\left(\frac{32}{5} - \frac{1}{8}\right) - \left(\frac{1}{5} - \frac{1}{4}\right)\right] = 2\pi\left(\frac{31}{5} + \frac{1}{8}\right) = \frac{2\pi}{40}(8\cdot 31 + 5) = \frac{253\pi}{20}$

24. $y = \cos x\ \Rightarrow\ \frac{dy}{dx} = -\sin x\ \Rightarrow\ \left(\frac{dy}{dx}\right)^2 = \sin^2 x\ \Rightarrow\ S = 2\pi\int_{-\pi/2}^{\pi/2}(\cos x)\sqrt{1 + \sin^2 x}\,dx$

25. $y = \sqrt{a^2 - x^2}\ \Rightarrow\ \frac{dy}{dx} = \frac{1}{2}\left(a^2 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{a^2 - x^2}}\ \Rightarrow\ \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{\left(a^2 - x^2\right)}$

$\Rightarrow\ S = 2\pi\int_{-a}^a\sqrt{a^2 - x^2}\,\sqrt{1 + \frac{x^2}{\left(a^2 - x^2\right)}}\,dx = 2\pi\int_{-a}^a\sqrt{\left(a^2 - x^2\right) + x^2}\,dx = 2\pi\int_{-a}^a a\,dx = 2\pi a[x]_{-a}^a$

$= 2\pi a[a - (-a)] = (2\pi a)(2a) = 4\pi a^2$

26. $y = \frac{r}{h}x \Rightarrow \frac{dy}{dx} = \frac{r}{h} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{h^2} \Rightarrow S = 2\pi \int_0^h \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\int_0^h \frac{r}{h}x\sqrt{\frac{h^2+r^2}{h^2}}\,dx$
$= \frac{2\pi r}{h}\sqrt{\frac{h^2+r^2}{h^2}}\int_0^h x\,dx = \frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left[\frac{x^2}{2}\right]_0^h = \frac{2\pi r}{h^2}\sqrt{h^2+r^2}\left(\frac{h^2}{2}\right) = \pi r\sqrt{h^2+r^2}$

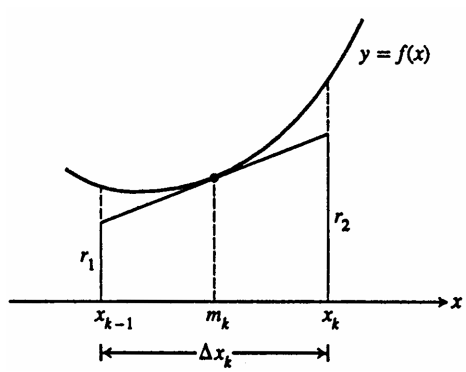
27. The area of the surface of one wok is $S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$. Now, $x^2+y^2 = 16^2 \Rightarrow x = \sqrt{16^2-y^2}$
$\Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{16^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{16^2-y^2}$; $S = \int_{-16}^{-7} 2\pi\sqrt{16^2-y^2}\sqrt{1+\frac{y^2}{16^2-y^2}}\,dy = 2\pi\int_{-16}^{-7}\sqrt{(16^2-y^2)+y^2}\,dy$
$= 2\pi\int_{-16}^{-7} 16\,dy = 32\pi\cdot 9 = 288\pi \approx 904.78\text{ cm}^2$. The enamel needed to cover one surface of one wok is
$V = S\cdot 0.5\text{ mm} = S\cdot 0.05\text{ cm} = (904.78)(0.05)\text{ cm}^3 = 45.24\text{ cm}^3$. For 5000 woks, we need
$5000\cdot V = 5000\cdot 45.24\text{ cm}^3 = (5)(45.24)\text{L} = 226.2\text{L} \Rightarrow$ 226.2 liters of each color are needed.

28. $y = \sqrt{r^2-x^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\,\frac{2x}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{x^2}{r^2-x^2}$; $S = 2\pi\int_a^{a+h}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$
$= 2\pi\int_a^{a+h}\sqrt{(r^2-x^2)+x^2}\,dx = 2\pi r\int_a^{a+h}dx = 2\pi rh$, which is independent of a.

29. $y = \sqrt{R^2-x^2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\,\frac{2x}{\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{x^2}{R^2-x^2}$; $S = 2\pi\int_a^{a+h}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx$
$= 2\pi\int_a^{a+h}\sqrt{(R^2-x^2)+x^2}\,dx = 2\pi R\int_a^{a+h}dx = 2\pi Rh$

30. (a) $x^2+y^2 = 45^2 \Rightarrow x = \sqrt{45^2-y^2} \Rightarrow \frac{dx}{dy} = \frac{-y}{\sqrt{45^2-y^2}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{45^2-y^2}$;
$S = \int_{-22.5}^{45} 2\pi\sqrt{45^2-y^2}\sqrt{1+\frac{y^2}{45^2-y^2}}\,dy = 2\pi\int_{-22.5}^{45}\sqrt{(45^2-y^2)+y^2}\,dy = 2\pi\cdot 45\int_{-22.5}^{45}dy$
$= (2\pi)(45)(67.5) = 6075\pi$ square feet
(b) 19,085 square feet

31. (a) An equation of the tangent line segment is (see figure) $y = f(m_k) + f'(m_k)(x-m_k)$.
When $x = x_{k-1}$ we have
$r_1 = f(m_k) + f'(m_k)(x_{k-1}-m_k)$
$= f(m_k) + f'(m_k)\left(-\frac{\Delta x_k}{2}\right) = f(m_k) - f'(m_k)\frac{\Delta x_k}{2}$;
when $x = x_k$ we have
$r_2 = f(m_k) + f'(m_k)(x_k-m_k)$
$= f(m_k) + f'(m_k)\frac{\Delta x_k}{2}$;



(b) $L_k^2 = (\Delta x_k)^2 + (r_2-r_1)^2$
$= (\Delta x_k)^2 + \left[f'(m_k)\frac{\Delta x_k}{2} - \left(-f'(m_k)\frac{\Delta x_k}{2}\right)\right]^2$
$= (\Delta x_k)^2 + [f'(m_k)\Delta x_k]^2 \Rightarrow L_k = \sqrt{(\Delta x_k)^2+[f'(m_k)\Delta x_k]^2}$, as claimed
(c) From geometry it is a fact that the lateral surface area of the frustum obtained by revolving the tangent line segment about the x-axis is given by $\Delta S_k = \pi(r_1+r_2)L_k = \pi[2f(m_k)]\sqrt{(\Delta x_k)^2+[f'(m_k)\Delta x_k]^2}$ using parts (a) and (b) above. Thus, $\Delta S_k = 2\pi f(m_k)\sqrt{1+[f'(m_k)]^2}\,\Delta x_k$.
(d) $S = \lim_{n\to\infty}\sum_{k=1}^{n}\Delta S_k = \lim_{n\to\infty}\sum_{k=1}^{n} 2\pi f(m_k)\sqrt{1+[f'(m_k)]^2}\,\Delta x_k = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

32. $y = \left(1-x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(1-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\left(1-x^{2/3}\right)^{1/2}}{x^{1/3}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}} = \frac{1}{x^{2/3}} - 1$
$\Rightarrow S = 2\int_0^1 2\pi\left(1-x^{2/3}\right)^{3/2}\sqrt{1+\left(\frac{1}{x^{2/3}}-1\right)}\,dx = 4\pi\int_0^1\left(1-x^{2/3}\right)^{3/2}\sqrt{x^{-2/3}}\,dx$

$= 4\pi \int_0^1 \left(1 - x^{2/3}\right)^{3/2} x^{-1/3}\, dx$; $\left[u = 1 - x^{2/3} \Rightarrow du = -\frac{2}{3} x^{-1/3}\, dx \Rightarrow -\frac{3}{2}\, du = x^{-1/3}\, dx;\right.$
$\left. x = 0 \Rightarrow u = 1, x = 1 \Rightarrow u = 0\right] \rightarrow S = 4\pi \int_1^0 u^{3/2} \left(-\frac{3}{2}\, du\right) = -6\pi \left[\frac{2}{5} u^{5/2}\right]_1^0 = -6\pi \left(0 - \frac{2}{5}\right) = \frac{12\pi}{5}$

6.5 WORK AND FLUID FORCES

1. The force required to stretch the spring from its natural length of 2 m to a length of 5 m is $F(x) = kx$. The work done by F is $W = \int_0^3 F(x)\, dx = k \int_0^3 x\, dx = \frac{k}{2}\left[x^2\right]_0^3 = \frac{9k}{2}$. This work is equal to 1800 J $\Rightarrow \frac{9}{2} k = 1800 \Rightarrow k = 400$ N/m

2. (a) We find the force constant from Hooke's Law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{800}{4} = 200$ lb/in.
   (b) The work done to stretch the spring 2 inches beyond its natural length is $W = \int_0^2 kx\, dx = 200 \int_0^2 x\, dx = 200 \left[\frac{x^2}{2}\right]_0^2$ $= 200(2 - 0) = 400\ \text{in} \cdot \text{lb} = 33.3\ \text{ft} \cdot \text{lb}$
   (c) We substitute $F = 1600$ into the equation $F = 200x$ to find $1600 = 200x \Rightarrow x = 8$ in.

3. We find the force constant from Hooke's law: $F = kx$. A force of 2 N stretches the spring to 0.02 m $\Rightarrow 2 = k \cdot (0.02)$ $\Rightarrow k = 100\ \frac{N}{m}$. The force of 4 N will stretch the rubber band y m, where $F = ky \Rightarrow y = \frac{F}{k} \Rightarrow y = \frac{4N}{100\,\frac{N}{m}} \Rightarrow y = 0.04$ m $= 4$ cm. The work done to stretch the rubber band 0.04 m is $W = \int_0^{0.04} kx\, dx = 100 \int_0^{0.04} x\, dx = 100 \left[\frac{x^2}{2}\right]_0^{0.04}$ $= \frac{(100)(0.04)^2}{2} = 0.08$ J

4. We find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} \Rightarrow k = \frac{90}{1} \Rightarrow k = 90\ \frac{N}{m}$. The work done to stretch the spring 5 m beyond its natural length is $W = \int_0^5 kx\, dx = 90 \int_0^5 x\, dx = 90 \left[\frac{x^2}{2}\right]_0^5 = (90)\left(\frac{25}{2}\right) = 1125$ J

5. (a) We find the spring's constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{21{,}714}{8-5} = \frac{21{,}714}{3} \Rightarrow k = 7238\ \frac{\text{lb}}{\text{in}}$
   (b) The work done to compress the assembly the first half inch is $W = \int_0^{0.5} kx\, dx = 7238 \int_0^{0.5} x\, dx = 7238 \left[\frac{x^2}{2}\right]_0^{0.5}$ $= (7238)\, \frac{(0.5)^2}{2} = \frac{(7238)(0.25)}{2} \approx 905\ \text{in} \cdot \text{lb}$. The work done to compress the assembly the second half inch is: $W = \int_{0.5}^{1.0} kx\, dx = 7238 \int_{0.5}^{1.0} x\, dx = 7238 \left[\frac{x^2}{2}\right]_{0.5}^{1.0} = \frac{7238}{2}\left[1 - (0.5)^2\right] = \frac{(7238)(0.75)}{2} \approx 2714\ \text{in} \cdot \text{lb}$

6. First, we find the force constant from Hooke's law: $F = kx \Rightarrow k = \frac{F}{x} = \frac{150}{\left(\frac{1}{16}\right)} = 16 \cdot 150 = 2{,}400\ \frac{\text{lb}}{\text{in}}$. If someone compresses the scale $x = \frac{1}{8}$ in, he/she must weigh $F = kx = 2{,}400\left(\frac{1}{8}\right) = 300$ lb. The work done to compress the scale this far is $W = \int_0^{1/8} kx\, dx = 2400 \left[\frac{x^2}{2}\right]_0^{1/8} = \frac{2400}{2 \cdot 64} = 18.75\ \text{lb} \cdot \text{in.} = \frac{25}{16}\ \text{ft} \cdot \text{lb}$

7. The force required to haul up the rope is equal to the rope's weight, which varies steadily and is proportional to x, the length of the rope still hanging: $F(x) = 0.624x$. The work done is: $W = \int_0^{50} F(x)\, dx = \int_0^{50} 0.624x\, dx = 0.624 \left[\frac{x^2}{2}\right]_0^{50}$ $= 780$ J

8. The weight of sand decreases steadily by 72 lb over the 18 ft, at 4 lb/ft. So the weight of sand when the bag is x ft off the ground is $F(x) = 144 - 4x$. The work done is: $W = \int_a^b F(x)\, dx = \int_0^{18} (144 - 4x)dx = \left[144x - 2x^2\right]_0^{18} = 1944\ \text{ft} \cdot \text{lb}$

9. The force required to lift the cable is equal to the weight of the cable paid out: $F(x) = (4.5)(180 - x)$ where x is the position of the car off the first floor. The work done is: $W = \int_0^{180} F(x)\, dx = 4.5 \int_0^{180} (180 - x)\, dx$

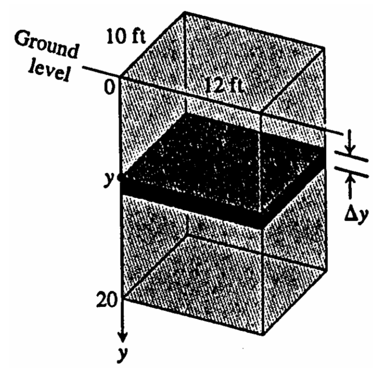
$= 4.5\left[180x - \frac{x^2}{2}\right]_0^{180} = 4.5\left(180^2 - \frac{180^2}{2}\right) = \frac{4.5 \cdot 180^2}{2} = 72{,}900 \text{ ft} \cdot \text{lb}$

10. Since the force is acting toward the origin, it acts opposite to the positive x-direction. Thus $F(x) = -\frac{k}{x^2}$. The work done is $W = \int_a^b -\frac{k}{x^2}\,dx = k\int_a^b -\frac{1}{x^2}\,dx = k\left[\frac{1}{x}\right]_a^b = k\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{k(a-b)}{ab}$

11. Let r = the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(20 - x)$, the distance the bucket is being raised. The leakage rate of the water is 0.8 lb/ft raised and the weight of the water in the bucket is $F = 0.8(20 - x)$. So:
$W = \int_0^{20} 0.8(20 - x)\,dx = 0.8\left[20x - \frac{x^2}{2}\right]_0^{20} = 160 \text{ ft} \cdot \text{lb}$.

12. Let r = the constant rate of leakage. Since the bucket is leaking at a constant rate and the bucket is rising at a constant rate, the amount of water in the bucket is proportional to $(20 - x)$, the distance the bucket is being raised. The leakage rate of the water is 2 lb/ft raised and the weight of the water in the bucket is $F = 2(20 - x)$. So:
$W = \int_0^{20} 2(20 - x)\,dx = 2\left[20x - \frac{x^2}{2}\right]_0^{20} = 400 \text{ ft} \cdot \text{lb}$.
Note that since the force in Exercise 12 is 2.5 times the force in Exercise 11 at each elevation, the total work is also 2.5 times as great.

13. We will use the coordinate system given.



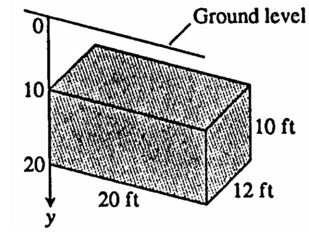
(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (10)(12)\,\Delta y = 120\,\Delta y \text{ ft}^3$. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 120\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force × distance $= 62.4 \cdot 120 \cdot y \cdot \Delta y$ ft · lb. The work it takes to lift all the water is approximately $W \approx \sum_0^{20} \Delta W$ $= \sum_0^{20} 62.4 \cdot 120y \cdot \Delta y$ ft · lb. This is a Riemann sum for the function $62.4 \cdot 120y$ over the interval $0 \le y \le 20$. The work of pumping the tank empty is the limit of these sums:
$W = \int_0^{20} 62.4 \cdot 120y\,dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{20} = (62.4)(120)\left(\frac{400}{2}\right) = (62.4)(120)(200) = 1{,}497{,}600 \text{ ft} \cdot \text{lb}$

(b) The time t it takes to empty the full tank with $\left(\frac{5}{11}\right)$–hp motor is $t = \frac{W}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{1{,}497{,}600 \text{ ft}\cdot\text{lb}}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = 5990.4 \text{ sec} = 1.664 \text{ hr}$
$\Rightarrow t \approx 1$ hr and 40 min

(c) Following all the steps of part (a), we find that the work it takes to lower the water level 10 ft is
$W = \int_0^{10} 62.4 \cdot 120y\,dy = (62.4)(120)\left[\frac{y^2}{2}\right]_0^{10} = (62.4)(120)\left(\frac{100}{2}\right) = 374{,}400 \text{ ft} \cdot \text{lb}$ and the time is $t = \frac{W}{250\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}}$
$= 1497.6 \text{ sec} = 0.416 \text{ hr} \approx 25 \text{ min}$

(d) In a location where water weighs $62.26\,\frac{\text{lb}}{\text{ft}^3}$:
a) $W = (62.26)(24{,}000) = 1{,}494{,}240$ ft · lb.
b) $t = \frac{1{,}494{,}240}{250} = 5976.96 \text{ sec} \approx 1.660 \text{ hr} \Rightarrow t \approx 1$ hr and 40 min
In a location where water weighs $62.59\,\frac{\text{lb}}{\text{ft}^3}$
a) $W = (62.59)(24{,}000) = 1{,}502{,}160$ ft · lb
b) $t = \frac{1{,}502{,}160}{250} = 6008.64 \text{ sec} \approx 1.669 \text{ hr} \Rightarrow t \approx 1$ hr and 40.1 min

14. We will use the coordinate system given.



(a) The typical slab between the planes at y and $y + \Delta y$ has a volume of $\Delta V = (20)(12)\,\Delta y = 240\,\Delta y\ \text{ft}^3$. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 240\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force × distance $= 62.4 \cdot 240 \cdot y \cdot \Delta y$ ft · lb. The work it takes to lift all the water is approximately $W \approx \sum_{10}^{20} \Delta W = \sum_{10}^{20} 62.4 \cdot 240y \cdot \Delta y$ ft · lb. This is a Riemann sum for the function $62.4 \cdot 240y$ over the interval $10 \le y \le 20$. The work it takes to empty the cistern is the limit of these sums: $W = \int_{10}^{20} 62.4 \cdot 240y\, dy$ $= (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{20} = (62.4)(240)(200 - 50) = (62.4)(240)(150) = 2{,}246{,}400$ ft · lb

(b) $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{2{,}246{,}400\ \text{ft}\cdot\text{lb}}{275} \approx 8168.73$ sec ≈ 2.27 hours $\approx$ 2 hr and 16.1 min

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is $W = \int_{10}^{15} 62.4 \cdot 240y\, dy = (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{15} = (62.4)(240)\left(\frac{225}{2} - \frac{100}{2}\right) = (62.4)(240)\left(\frac{125}{2}\right) = 936{,}000$ ft. Then the time is $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{936{,}000}{275} \approx 3403.64$ sec ≈ 56.7 min

(d) In a location where water weighs $62.26\ \frac{\text{lb}}{\text{ft}^3}$:

a) $W = (62.26)(240)(150) = 2{,}241{,}360$ ft · lb.

b) $t = \frac{2{,}241{,}360}{275} = 8150.40$ sec $= 2.264$ hours $\approx$ 2 hr and 15.8 min

c) $W = (62.26)(240)\left(\frac{125}{2}\right) = 933{,}900$ ft · lb; $t = \frac{933{,}900}{275} = 3396$ sec ≈ 0.94 hours ≈ 56.6 min

In a location where water weighs $62.59\ \frac{\text{lb}}{\text{ft}^3}$

a) $W = (62.59)(240)(150) = 2{,}253{,}240$ ft · lb.

b) $t = \frac{2{,}253{,}240}{275} = 8193.60$ sec $= 2.276$ hours $\approx$ 2 hr and 16.56 min

c) $W = (62.59)(240)\left(\frac{125}{2}\right) = 938{,}850$ ft · lb; $t = \frac{938{,}850}{275} \approx 3414$ sec ≈ 0.95 hours ≈ 56.9 min

15. The slab is a disk of area $\pi x^2 = \pi\left(\frac{y}{2}\right)^2$, thickness $\triangle y$, and height below the top of the tank $(10 - y)$. So the work to pump the oil in this slab, $\triangle W$, is $57(10 - y)\pi\left(\frac{y}{2}\right)^2$. The work to pump all the oil to the top of the tank is $W = \int_0^{10} \frac{57\pi}{4}(10y^2 - y^3)dy = \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^{10} = 11{,}875\pi$ ft · lb $\approx 37{,}306$ ft · lb.

16. Each slab of oil is to be pumped to a height of 14 ft. So the work to pump a slab is $(14 - y)(\pi)\left(\frac{y}{2}\right)^2$ and since the tank is half full and the volume of the original cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5^2)(10) = \frac{250\pi}{3}\ \text{ft}^3$, half the volume $= \frac{250\pi}{6}\ \text{ft}^3$, and with half the volume the cone is filled to a height y, $\frac{250\pi}{6} = \frac{1}{3}\pi\frac{y^2}{4}y \Rightarrow y = \sqrt[3]{500}$ ft. So $W = \int_0^{\sqrt[3]{500}} \frac{57\pi}{4}(14y^2 - y^3)\, dy$ $= \frac{57\pi}{4}\left[\frac{14y^3}{3} - \frac{y^4}{4}\right]_0^{\sqrt[3]{500}} \approx 60{,}042$ ft · lb.

17. The typical slab between the planes at y and and $y + \Delta y$ has a volume of $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{20}{2}\right)^2 \Delta y$ $= \pi \cdot 100\ \Delta y\ \text{ft}^3$. The force F required to lift the slab is equal to its weight: $F = 51.2\,\Delta V = 51.2 \cdot 100\pi\,\Delta y$ lb $\Rightarrow F = 5120\pi\,\Delta y$ lb. The distance through which F must act is about $(30 - y)$ ft. The work it takes to lift all the kerosene is approximately $W \approx \sum_0^{30} \Delta W = \sum_0^{30} 5120\pi(30 - y)\,\Delta y$ ft · lb which is a Riemann sum. The work to pump the tank dry is the limit of these sums: $W = \int_0^{30} 5120\pi(30 - y)\, dy = 5120\pi\left[30y - \frac{y^2}{2}\right]_0^{30} = 5120\pi\left(\frac{900}{2}\right) = (5120)(450\pi)$ $\approx 7{,}238{,}229.48$ ft · lb

18. (a) Follow all the steps of Example 5 but make the substitution of 64.5 $\frac{\text{lb}}{\text{ft}^3}$ for 57 $\frac{\text{lb}}{\text{ft}^3}$. Then,

$W = \int_0^8 \frac{64.5\pi}{4}(10-y)y^2\,dy = \frac{64.5\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{64.5\pi}{4}\left(\frac{10\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{64.5\pi}{4}\right)(8^3)\left(\frac{10}{3} - 2\right)$

$= \frac{64.5\pi\cdot 8^3}{3} = 21.5\pi\cdot 8^3 \approx 34{,}582.65$ ft · lb

(b) Exactly as done in Example 5 but change the distance through which F acts to distance $\approx (13-y)$ ft. Then

$W = \int_0^8 \frac{57\pi}{4}(13-y)y^2\,dy = \frac{57\pi}{4}\left[\frac{13y^3}{3} - \frac{y^4}{4}\right]_0^8 = \frac{57\pi}{4}\left(\frac{13\cdot 8^3}{3} - \frac{8^4}{4}\right) = \left(\frac{57\pi}{4}\right)(8^3)\left(\frac{13}{3} - 2\right) = \frac{57\pi\cdot 8^3\cdot 7}{3\cdot 4}$

$= (19\pi)(8^2)(7)(2) \approx 53{,}482.5$ ft · lb

19. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\sqrt{y}\right)^2\Delta y$ ft^3. The force F(y) required to lift this slab is equal to its weight: $F(y) = 73\cdot\Delta V = 73\pi\left(\sqrt{y}\right)^2\Delta y = 73\pi\, y\,\Delta y$ lb. The distance through which F(y) must act to lift the slab to the top of the reservoir is about $(4-y)$ ft, so the work done is approximately $\Delta W \approx 73\pi\, y\,(4-y)\Delta y$ ft · lb. The work done lifting all the slabs from y = 0 ft to y = 4 ft is approximately $W \approx \sum_{k=0}^{n} 73\pi\, y_k\,(4-y_k)\Delta y$ ft · lb. Taking the limit of these Riemann sums as $n \to \infty$, we get

$W = \int_0^4 73\pi\, y\,(4-y)dy = 73\pi\int_0^4 (4y - y^2)dy = 73\pi\left[2y^2 - \frac{1}{3}y^3\right]_0^4 = 73\pi\left(32 - \frac{64}{3}\right) = \frac{2336\pi}{3}$ ft · lb.

20. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = (\text{length})(\text{width})(\text{thickness})$ $= \left(2\sqrt{25-y^2}\right)(10)\,\Delta y$ ft^3. The force F(y) required to lift this slab is equal to its weight: $F(y) = 53\cdot\Delta V$ $= 53\left(2\sqrt{25-y^2}\right)(10)\,\Delta y = 1060\sqrt{25-y^2}\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 15 m above the top of the reservoir is about $(20-y)$ ft, so the work done is approximately $\Delta W \approx 1060\sqrt{25-y^2}(20-y)\Delta y$ ft · lb. The work done lifting all the slabs from $y=-5$ ft to $y=5$ ft is approximately $W \approx \sum_{k=0}^{n} 1060\sqrt{25-y_k^2}\,(20-y_k)\Delta y$ ft · lb. Taking the limit of these Riemann sums as $n\to\infty$, we get

$W = \int_{-5}^{5} 1060\sqrt{25-y^2}(20-y)dy = 1060\int_{-5}^{5}(20-y)\sqrt{25-y^2}dy = 1060\left[\int_{-5}^{5} 20\sqrt{25-y^2}dy - \int_{-5}^{5} y\sqrt{25-y^2}dy\right]$

To evaluate the first integral, we use we can interpret $\int_{-5}^{5}\sqrt{25-y^2}dy$ as the area of the semicircle whose radius is 5, thus $\int_{-5}^{5} 20\sqrt{25-y^2}dy = 20\int_{-5}^{5}\sqrt{25-y^2}dy = 20\left[\frac{1}{2}\pi(5)^2\right] = 250\pi$. To evaluate the second integral let $u = 25 - y^2$ $\Rightarrow du = -2y\,dy$; $y=-5 \Rightarrow u=0$, $y=5 \Rightarrow u=0$, thus $\int_{-5}^{5} y\sqrt{25-y^2}dy = -\frac{1}{2}\int_0^0 \sqrt{u}\,du = 0$. Thus,

$1060\left[\int_{-5}^{5} 20\sqrt{25-y^2}dy - \int_{-5}^{5} y\sqrt{25-y^2}dy\right] = 1060(250\pi - 0) = 265000\pi \approx 832522$ ft · lb.

21. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{25-y^2}\right)^2\Delta y$ m^3. The force F(y) required to lift this slab is equal to its weight: $F(y) = 9800\cdot\Delta V$ $= 9800\pi\left(\sqrt{25-y^2}\right)^2\Delta y = 9800\pi(25-y^2)\,\Delta y$ N. The distance through which F(y) must act to lift the slab to the level of 4 m above the top of the reservoir is about $(4-y)$ m, so the work done is approximately $\Delta W \approx 9800\pi(25-y^2)(4-y)\,\Delta y$ N · m. The work done lifting all the slabs from $y=-5$ m to $y=0$ m is approximately $W \approx \sum_{-5}^{0} 9800\pi(25-y^2)(4-y)\,\Delta y$ N · m. Taking the limit of these Riemann sums, we get

$W = \int_{-5}^{0} 9800\pi(25-y^2)(4-y)\,dy = 9800\pi\int_{-5}^{0}(100 - 25y - 4y^2 + y^3)\,dy = 9800\pi\left[100y - \frac{25}{2}y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_{-5}^{0}$

$= -9800\pi\left(-500 - \frac{25\cdot 25}{2} + \frac{4}{3}\cdot 125 + \frac{625}{4}\right) \approx 15{,}073{,}099.75$ J

22. The typical slab between the planes at y and y+Δy has a volume of about $\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\sqrt{100-y^2}\right)^2\Delta y = \pi(100-y^2)\,\Delta y$ ft^3. The force is $F(y) = \frac{56\text{ lb}}{\text{ft}^3}\cdot\Delta V = 56\pi(100-y^2)\,\Delta y$ lb. The distance through which F(y) must act to lift the slab to the level of 2 ft above the top of the tank is about

$(12 - y)$ ft, so the work done is $\Delta W \approx 56\pi\,(100 - y^2)\,(12 - y)\,\Delta y$ lb · ft. The work done lifting all the slabs from $y = 0$ ft to $y = 10$ ft is approximately $W \approx \sum_{0}^{10} 56\pi\,(100 - y^2)\,(12 - y)\,\Delta y$ lb · ft. Taking the limit of these Riemann sums, we get $W = \int_0^{10} 56\pi\,(100 - y^2)\,(12 - y)\,dy = 56\pi \int_0^{10} (100 - y^2)\,(12 - y)\,dy$

$= 56\pi \int_0^{10} (1200 - 100y - 12y^2 + y^3)\,dy = 56\pi \left[1200y - \frac{100y^2}{2} - \frac{12y^3}{3} + \frac{y^4}{4}\right]_0^{10}$

$= 56\pi \left(12{,}000 - \frac{10{,}000}{2} - 4 \cdot 1000 + \frac{10{,}000}{4}\right) = (56\pi)\left(12 - 5 - 4 + \frac{5}{2}\right)(1000) \approx 967{,}611$ ft · lb.

It would cost (0.5)(967,611) = 483,805¢ = \$4838.05. Yes, you can afford to hire the firm.

23. $F = m\,\frac{dv}{dt} = mv\,\frac{dv}{dx}$ by the chain rule $\Rightarrow$ $W = \int_{x_1}^{x_2} mv\,\frac{dv}{dx}\,dx = m\int_{x_1}^{x_2} \left(v\,\frac{dv}{dx}\right) dx = m\left[\frac{1}{2}\,v^2(x)\right]_{x_1}^{x_2}$

$= \frac{1}{2}\,m\left[v^2(x_2) - v^2(x_1)\right] = \frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2$, as claimed.

24. weight $= 2$ oz $= \frac{2}{16}$ lb; mass $= \frac{\text{weight}}{32} = \frac{\frac{1}{8}}{32} = \frac{1}{256}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{256}\text{ slugs}\right)(160\text{ ft/sec})^2 \approx 50$ ft · lb

25. 90 mph $= \frac{90\text{ mi}}{1\text{ hr}} \cdot \frac{1\text{ hr}}{60\text{ min}} \cdot \frac{1\text{ min}}{60\text{ sec}} \cdot \frac{5280\text{ ft}}{1\text{ mi}} = 132$ ft/sec; $m = \frac{0.3125\text{ lb}}{32\text{ ft/sec}^2} = \frac{0.3125}{32}$ slugs;
$W = \left(\frac{1}{2}\right)\left(\frac{0.3125\text{ lb}}{32\text{ ft/sec}^2}\right)(132\text{ ft/sec})^2 \approx 85.1$ ft · lb

26. weight $= 1.6$ oz $= 0.1$ lb $\Rightarrow$ $m = \frac{0.1\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{320}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{1}{320}\text{ slugs}\right)(280\text{ ft/sec})^2 = 122.5$ ft · lb

27. $v_1 = 0$ mph $= 0\,\frac{\text{ft}}{\text{sec}}$, $v_2 = 153$ mph $= 224.4\,\frac{\text{ft}}{\text{sec}}$; 2 oz $= 0.125$ lb $\Rightarrow m = \frac{0.125\text{ lb}}{32\text{ ft/sec}^2} = \frac{1}{256}$ slugs;
$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2 = \frac{1}{2}\left(\frac{1}{256}\right)(224.4)^2 - \frac{1}{2}\left(\frac{1}{256}\right)(0)^2 = 98.35$ ft-lb.

28. weight $= 6.5$ oz $= \frac{6.5}{16}$ lb $\Rightarrow$ $m = \frac{6.5}{(16)(32)}$ slugs; $W = \left(\frac{1}{2}\right)\left(\frac{6.5}{(16)(32)}\text{ slugs}\right)(132\text{ ft/sec})^2 \approx 110.6$ ft · lb

29. We imagine the milkshake divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 7]$. The typical slab between the planes at y and $y + \Delta y$ has a volume of about$\Delta V = \pi(\text{radius})^2(\text{thickness})$ $= \pi\left(\frac{y + 17.5}{14}\right)^2 \Delta y$ in^3. The force $F(y)$ required to lift this slab is equal to its weight: $F(y) = \frac{4}{9}\;\Delta V = \frac{4\pi}{9}\left(\frac{y + 17.5}{14}\right)^2 \Delta y$ oz. The distance through which $F(y)$ must act to lift this slab to the level of 1 inch above the top is about $(8 - y)$ in. The work done lifting the slab is about $\Delta W = \left(\frac{4\pi}{9}\right)\frac{(y + 17.5)^2}{14^2}(8 - y)\Delta y$ in · oz. The work done lifting all the slabs from $y = 0$ to $y = 7$ is approximately $W = \sum_{0}^{7} \frac{4\pi}{9 \cdot 14^2}(y + 17.5)^2(8 - y)\,\Delta y$ in · oz which is a Riemann sum. The work is the limit of these sums as the norm of the partition goes to zero: $W = \int_0^7 \frac{4\pi}{9 \cdot 14^2}(y + 17.5)^2(8 - y)dy$

$= \frac{4\pi}{9 \cdot 14^2} \int_0^7 (2450 - 26.25y - 27y^2 - y^3)dy = \frac{4\pi}{9 \cdot 14^2}\left[-\frac{y^4}{4} - 9y^3 - \frac{26.25}{2}\;y^2 + 2450y\right]_0^7$

$= \frac{4\pi}{9 \cdot 14^2}\left[-\frac{7^4}{4} - 9 \cdot 7^3 - \frac{26.25}{2} \cdot 7^2 + 2450 \cdot 7\right] \approx 91.32$ in · oz

30. $\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000\,MG}{r^2}\,dr = 1000\,MG \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{dr}{r^2} = 1000\,MG\left[-\frac{1}{r}\right]_{6{,}370{,}000}^{35{,}780{,}000}$

$= (1000)\,(5.975 \cdot 10^{24})\,(6.672 \cdot 10^{-11})\left(\frac{1}{6{,}370{,}000} - \frac{1}{35{,}780{,}000}\right) \approx 5.144 \times 10^{10}$ J

31. To find the width of the plate at a typical depth y, we first find an equation for the line of the plate's right-hand edge: $y = x - 5$. If we let x denote the width of the right-hand half of the triangle at depth y, then $x = 5 + y$ and the total width is $L(y) = 2x = 2(5 + y)$. The depth of the strip is $(-y)$. The force exerted by the water against one side of the plate is therefore $F = \int_{-5}^{-2} w(-y) \cdot L(y)\,dy = \int_{-5}^{-2} 62.4 \cdot (-y) \cdot 2(5 + y)\,dy$

$= 124.8 \int_{-5}^{-2} (-5y - y^2)\, dy = 124.8 \left[-\frac{5}{2} y^2 - \frac{1}{3} y^3\right]_{-5}^{-2} = 124.8 \left[\left(-\frac{5}{2} \cdot 4 + \frac{1}{3} \cdot 8\right) - \left(-\frac{5}{2} \cdot 25 + \frac{1}{3} \cdot 125\right)\right]$
$= (124.8)\left(\frac{105}{2} - \frac{117}{3}\right) = (124.8)\left(\frac{315 - 234}{6}\right) = 1684.8$ lb

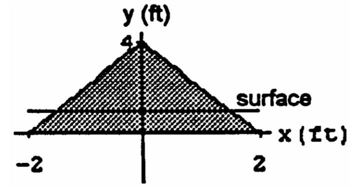
32. An equation for the line of the plate's right-hand edge is $y = x - 3 \Rightarrow x = y + 3$. Thus the total width is $L(y) = 2x = 2(y + 3)$. The depth of the strip is $(2 - y)$. The force exerted by the water is
$F = \int_{-3}^{0} w(2 - y)L(y)\, dy = \int_{-3}^{0} 62.4 \cdot (2 - y) \cdot 2(3 + y)\, dy = 124.8 \int_{-3}^{0} (6 - y - y^2)\, dy = 124.8 \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{0}$
$= (-124.8)\left(-18 - \frac{9}{2} + 9\right) = (-124.8)\left(-\frac{27}{2}\right) = 1684.8$ lb

33. (a) The width of the strip is $L(y) = 4$, the depth of the strip is $(10 - y) \Rightarrow F = \int_a^b w \cdot \binom{\text{strip}}{\text{depth}} F(y)dy$
$= \int_0^3 62.4(10 - y)(4)dy = 249.6 \int_0^3 (10 - y)dy = 249.6\left[10y - \frac{y^2}{2}\right]_0^3 = 249.6\left(30 - \frac{9}{2}\right) = 6364.8$ lb
(b) The width of the strip is $L(y) = 3$, the depth of the strip is $(10 - y) \Rightarrow F = \int_a^b w \cdot \binom{\text{strip}}{\text{depth}} F(y)dy$
$= \int_0^4 62.4(10 - y)(3)dy = 187.2 \int_0^4 (10 - y)dy = 187.2\left[10y - \frac{y^2}{2}\right]_0^4 = 187.2(40 - 8) = 5990.4$ lb

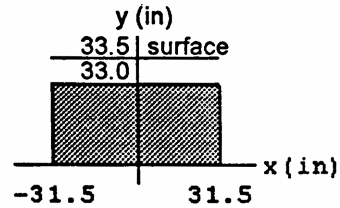
34. The width of the strip is $L(y) = 2\sqrt{25 - y^2}$, the depth of the strip is $(6 - y) \Rightarrow F = \int_a^b w \cdot \binom{\text{strip}}{\text{depth}} F(y)dy$
$= \int_0^5 62.4(6 - y)\left(2\sqrt{25 - y^2}\right)dy = 124.8 \int_0^5 (6 - y)\sqrt{25 - y^2}dy = 124.8\left[\int_0^5 6\sqrt{25 - y^2}dy - \int_0^5 y\sqrt{25 - y^2}dy\right]$
To evaluate the first integral, we use we can interpret $\int_0^5 \sqrt{25 - y^2}dy$ as the area of a quarter circle whose radius is 5, thus $\int_0^5 6\sqrt{25 - y^2}dy = 6\int_0^5 \sqrt{25 - y^2}dy = 6\left[\frac{1}{4}\pi(5)^2\right] = \frac{75\pi}{2}$. To evaluate the second integral let $u = 25 - y^2$
$\Rightarrow du = -2y\, dy; y = 0 \Rightarrow u = 25, y = 5 \Rightarrow u = 0$, thus $\int_0^5 y\sqrt{25 - y^2}dy = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du = \frac{1}{2}\int_0^{25} u^{1/2}\, du$
$= \frac{1}{3}\left[u^{3/2}\right]_0^{25} = \frac{125}{3}$. Thus, $124.8\left[\int_0^5 6\sqrt{25 - y^2}dy - \int_0^5 y\sqrt{25 - y^2}dy\right] = 124.8\left(\frac{75\pi}{2} - \frac{125}{3}\right) \approx 9502.7$ lb.

35. Using the coordinate system of Exercise 32, we find the equation for the line of the plate's right-hand edge to be $y = 2x - 4 \Rightarrow x = \frac{y+4}{2}$ and $L(y) = 2x = y + 4$. The depth of the strip is $(1 - y)$.
(a) $F = \int_{-4}^{0} w(1 - y)L(y)\, dy = \int_{-4}^{0} 62.4 \cdot (1 - y)(y + 4)\, dy = 62.4 \int_{-4}^{0} (4 - 3y - y^2)\, dy = 62.4 \left[4y - \frac{3y^2}{2} - \frac{y^3}{3}\right]_{-4}^{0}$
$= (-62.4)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = (-62.4)\left(-16 - 24 + \frac{64}{3}\right) = \frac{(-62.4)(-120 + 64)}{3} = 1164.8$ lb
(b) $F = (-64.0)\left[(-4)(4) - \frac{(3)(16)}{2} + \frac{64}{3}\right] = \frac{(-64.0)(-120 + 64)}{3} \approx 1194.7$ lb

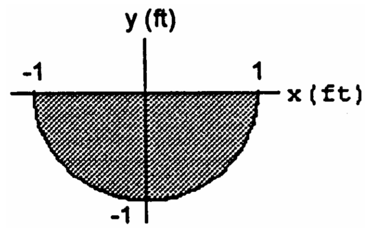
36. Using the coordinate system given, we find an equation for the line of the plate's right-hand edge to be $y = -2x + 4$ $\Rightarrow x = \frac{4-y}{2}$ and $L(y) = 2x = 4 - y$. The depth of the strip is $(1 - y) \Rightarrow F = \int_0^1 w(1 - y)(4 - y)\, dy$
$= 62.4 \int_0^1 (y^2 - 5y + 4)\, dy = 62.4 \left[\frac{y^3}{3} - \frac{5y^2}{2} + 4y\right]_0^1$
$= (62.4)\left(\frac{1}{3} - \frac{5}{2} + 4\right) = (62.4)\left(\frac{2 - 15 + 24}{6}\right) = \frac{(62.4)(11)}{6} = 114.4$ lb



37. Using the coordinate system given in the accompanying figure, we see that the total width is $L(y) = 63$ and the depth of the strip is $(33.5 - y) \Rightarrow F = \int_0^{33} w(33.5 - y)L(y)\,dy$
$= \int_0^{33} \frac{64}{12^3} \cdot (33.5 - y) \cdot 63\,dy = \left(\frac{64}{12^3}\right)(63)\int_0^{33} (33.5 - y)\,dy$
$= \left(\frac{64}{12^3}\right)(63)\left[33.5y - \frac{y^2}{2}\right]_0^{33} = \left(\frac{64 \cdot 63}{12^3}\right)\left[(33.5)(33) - \frac{33^2}{2}\right]$
$= \frac{(64)(63)(33)(67-33)}{(2)(12^3)} = 1309$ lb



38. Using the coordinate system given in the accompanying figure, we see that the right-hand edge is $x = \sqrt{1-y^2}$ so the total width is $L(y) = 2x = 2\sqrt{1-y^2}$ and the depth of the strip is $(-y)$. The force exerted by the water is therefore $F = \int_{-1}^{0} w \cdot (-y) \cdot 2\sqrt{1-y^2}\,dy$



$= 62.4\int_{-1}^{0} \sqrt{1-y^2}\,d\left(1-y^2\right) = 62.4\left[\frac{2}{3}\left(1-y^2\right)^{3/2}\right]_{-1}^{0} = (62.4)\left(\frac{2}{3}\right)(1-0) = 41.6$ lb

39. (a) $F = \left(62.4\,\frac{\text{lb}}{\text{ft}^3}\right)(8\text{ ft})(25\text{ ft}^2) = 12480$ lb

(b) The width of the strip is $L(y) = 5$, the depth of the strip is $(8 - y) \Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy$
$= \int_0^5 62.4(8-y)(5)dy = 312\int_0^5 (8-y)dy = 312\left[8y - \frac{y^2}{2}\right]_0^5 = 312\left(40 - \frac{25}{2}\right) = 8580$ lb

(c) The width of the strip is $L(y) = 5$, the depth of the strip is $(8 - y)$, the height of the strip is $\sqrt{2}\,dy$
$\Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy = \int_0^{5/\sqrt{2}} 62.4(8-y)(5)\sqrt{2}\,dy = 312\sqrt{2}\int_0^{5/\sqrt{2}} (8-y)dy = 312\sqrt{2}\left[8y - \frac{y^2}{2}\right]_0^{5/\sqrt{2}}$
$= 312\sqrt{2}\left(\frac{40}{\sqrt{2}} - \frac{25}{4}\right) = 9722.3$

40. The width of the strip is $L(y) = \frac{3}{4}\left(2\sqrt{3} - y\right)$, the depth of the strip is $(6 - y)$, the height of the strip is $\frac{2}{\sqrt{3}}dy$
$\Rightarrow F = \int_a^b w \cdot \left(\begin{smallmatrix}\text{strip}\\ \text{depth}\end{smallmatrix}\right)F(y)dy = \int_0^{2\sqrt{3}} 62.4(6-y) \cdot \frac{3}{4}\left(2\sqrt{3} - y\right)\frac{2}{\sqrt{3}}dy = \frac{93.6}{\sqrt{3}}\int_0^{2\sqrt{3}} \left(12\sqrt{3} - 6y - 2y\sqrt{3} + y^2\right)dy$
$= \frac{93.6}{\sqrt{3}}\left[12y\sqrt{3} - 3y^2 - y^2\sqrt{3} + \frac{y^3}{3}\right]_0^{2\sqrt{3}} = \frac{93.6}{\sqrt{3}}\left(72 - 36 - 12\sqrt{3} + 8\sqrt{3}\right) \approx 1571.04$ lb

41. The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

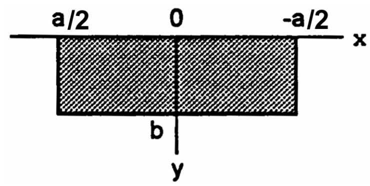
(a) The depth of the strip is $(2 - y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\,dy$
$= \int_0^1 50(2-y) \cdot 2\sqrt{y}\,dy = 100\int_0^1 (2-y)\sqrt{y}\,dy = 100\int_0^1 \left(2y^{1/2} - y^{3/2}\right)dy = 100\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1$
$= 100\left(\frac{4}{3} - \frac{2}{5}\right) = \left(\frac{100}{15}\right)(20 - 6) = 93.33$ lb

(b) We need to solve $160 = \int_0^1 w(H-y) \cdot 2\sqrt{y}\,dy$ for h. $160 = 100\left(\frac{2H}{3} - \frac{2}{5}\right) \Rightarrow H = 3$ ft.

42. Suppose that h is the maximum height. Using the coordinate system given in the text, we find an equation for the line of the end plate's right-hand edge is $y = \frac{5}{2}x \Rightarrow x = \frac{2}{5}y$. The total width is $L(y) = 2x = \frac{4}{5}y$ and the depth of the typical horizontal strip at level y is $(h - y)$. Then the force is $F = \int_0^h w(h-y)L(y)\,dy = F_{max}$, where $F_{max} = 6667$ lb. Hence, $F_{max} = w\int_0^h (h-y) \cdot \frac{4}{5}y\,dy = (62.4)\left(\frac{4}{5}\right)\int_0^h \left(hy - y^2\right)dy$
$= (62.4)\left(\frac{4}{5}\right)\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = (62.4)\left(\frac{4}{5}\right)\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = (62.4)\left(\frac{4}{5}\right)\left(\frac{1}{6}\right)h^3 = (10.4)\left(\frac{4}{5}\right)h^3 \Rightarrow h = \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{F_{max}}{10.4}\right)}$

$= \sqrt[3]{\left(\frac{5}{4}\right)\left(\frac{6667}{10.4}\right)} \approx 9.288$ ft. The volume of water which the tank can hold is $V = \frac{1}{2}$ (Base)(Height) $\cdot$ 30, where Height = h and $\frac{1}{2}$ (Base) $= \frac{2}{5}h \Rightarrow V = \left(\frac{2}{5}h^2\right)(30) = 12h^2 \approx 12(9.288)^2 \approx 1035$ ft^3.

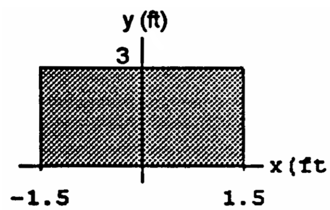
43. The pressure at level y is $p(y) = w \cdot y \Rightarrow$ the average pressure is $\overline{p} = \frac{1}{b}\int_0^b p(y)\,dy = \frac{1}{b}\int_0^b w \cdot y\,dy = \frac{1}{b}w\left[\frac{y^2}{2}\right]_0^b = \left(\frac{w}{b}\right)\left(\frac{b^2}{2}\right) = \frac{wb}{2}$. This is the pressure at level $\frac{b}{2}$, which is the pressure at the middle of the plate.



44. The force exerted by the fluid is $F = \int_0^b w(\text{depth})(\text{length})\,dy = \int_0^b w \cdot y \cdot a\,dy = (w \cdot a)\int_0^b y\,dy = (w \cdot a)\left[\frac{y^2}{2}\right]_0^b$ $= w\left(\frac{ab^2}{2}\right) = \left(\frac{wb}{2}\right)(ab) = \overline{p} \cdot \text{Area}$, where $\overline{p}$ is the average value of the pressure.

45. When the water reaches the top of the tank the force on the movable side is $\int_{-2}^{0}(62.4)\left(2\sqrt{4-y^2}\right)(-y)\,dy$ $= (62.4)\int_{-2}^{0}(4-y^2)^{1/2}(-2y)\,dy = (62.4)\left[\frac{2}{3}(4-y^2)^{3/2}\right]_{-2}^{0} = (62.4)\left(\frac{2}{3}\right)\left(4^{3/2}\right) = 332.8$ ft $\cdot$ lb. The force compressing the spring is $F = 100x$, so when the tank is full we have $332.8 = 100x \Rightarrow x \approx 3.33$ ft. Therefore the movable end does not reach the required 5 ft to allow drainage $\Rightarrow$ the tank will overflow.

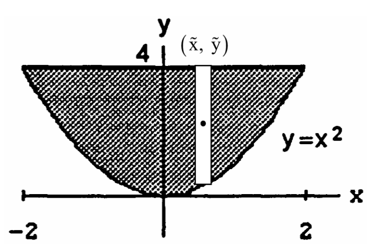
46. (a) Using the given coordinate system we see that the total width is $L(y) = 3$ and the depth of the strip is $(3 - y)$. Thus, $F = \int_0^3 w(3-y)L(y)\,dy = \int_0^3 (62.4)(3-y) \cdot 3\,dy$
$= (62.4)(3)\int_0^3 (3-y)\,dy = (62.4)(3)\left[3y - \frac{y^2}{2}\right]_0^3$
$= (62.4)(3)\left(9 - \frac{9}{2}\right) = (62.4)(3)\left(\frac{9}{2}\right) = 842.4$ lb



(b) Find a new water level Y such that $F_Y = (0.75)(842.4\text{ lb}) = 631.8$ lb. The new depth of the strip is $(Y - y)$ and Y is the new upper limit of integration. Thus, $F_Y = \int_0^Y w(Y-y)L(y)\,dy = 62.4\int_0^Y (Y-y) \cdot 3\,dy$ $= (62.4)(3)\int_0^Y (Y-y)\,dy = (62.4)(3)\left[Yy - \frac{y^2}{2}\right]_0^Y = (62.4)(3)\left(Y^2 - \frac{Y^2}{2}\right) = (62.4)(3)\left(\frac{Y^2}{2}\right)$. Therefore, $Y = \sqrt{\frac{2F_Y}{(62.4)(3)}} = \sqrt{\frac{1263.6}{187.2}} = \sqrt{6.75} \approx 2.598$ ft. So, $\Delta Y = 3 - Y \approx 3 - 2.598 \approx 0.402$ ft ≈ 4.8 in

6.6 MOMENTS AND CENTERS OF MASS

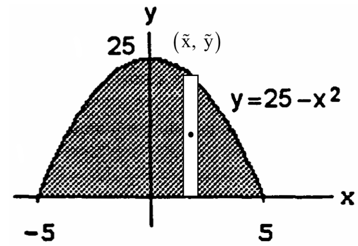
1. Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\overline{x} = 0$. It remains to find $\overline{y} = \frac{M_x}{M}$. We model the distribution of mass with *vertical* strips. The typical strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{x^2+4}{2}\right)$, length: $4 - x^2$, width: dx, area:



$dA = (4 - x^2)\,dx$, mass: $dm = \delta\,dA = \delta(4 - x^2)\,dx$. The moment of the strip about the x-axis is $\widetilde{y}\,dm = \left(\frac{x^2+4}{2}\right)\delta(4-x^2)\,dx = \frac{\delta}{2}(16 - x^4)\,dx$. The moment of the plate about the x-axis is $M_x = \int \widetilde{y}\,dm$ $= \int_{-2}^{2}\frac{\delta}{2}(16-x^4)\,dx = \frac{\delta}{2}\left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{\delta}{2}\left[\left(16 \cdot 2 - \frac{2^5}{5}\right) - \left(-16 \cdot 2 + \frac{2^5}{5}\right)\right] = \frac{\delta \cdot 2}{2}\left(32 - \frac{32}{5}\right) = \frac{128\delta}{5}$. The mass of the

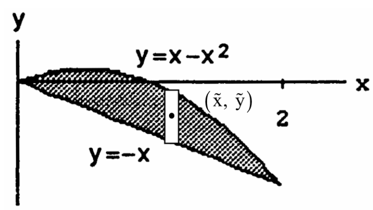
plate is $M = \int \delta\,(4 - x^2)\,dx = \delta\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 2\delta\left(8 - \frac{8}{3}\right) = \frac{32\delta}{3}$. Therefore $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{128\delta}{5}\right)}{\left(\frac{32\delta}{3}\right)} = \frac{12}{5}$. The plate's center of mass is the point $(\bar{x}, \bar{y}) = \left(0, \frac{12}{5}\right)$.

2. Applying the symmetry argument analogous to the one in Exercise 1, we find $\bar{x} = 0$. To find $\bar{y} = \frac{M_x}{M}$, we use the *vertical* strips technique. The typical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{25 - x^2}{2}\right)$, length: $25 - x^2$, width: dx, area: $dA = (25 - x^2)dx$, mass: $dm = \delta\,dA = \delta\,(25 - x^2)\,dx$. The moment of the strip about the x-axis is



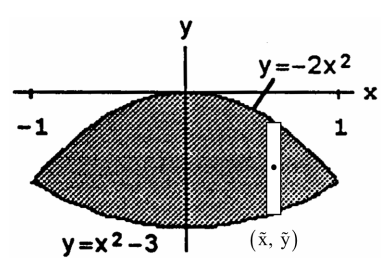
$\tilde{y}\,dm = \left(\frac{25 - x^2}{2}\right)\delta\,(25 - x^2)\,dx = \frac{\delta}{2}(25 - x^2)^2\,dx$. The moment of the plate about the x-axis is $M_x = \int \tilde{y}\,dm$
$= \int_{-5}^{5} \frac{\delta}{2}(25 - x^2)^2\,dx = \frac{\delta}{2}\int_{-5}^{5}(625 - 50x^2 + x^4)\,dx = \frac{\delta}{2}\left[625x - \frac{50}{3}x^3 + \frac{x^5}{5}\right]_{-5}^{5} = 2 \cdot \frac{\delta}{2}\left(625 \cdot 5 - \frac{50}{3} \cdot 5^3 + \frac{5^5}{5}\right)$
$= \delta \cdot 625\left(5 - \frac{10}{3} + 1\right) = \delta \cdot 625 \cdot \left(\frac{8}{3}\right)$. The mass of the plate is $M = \int dm = \int_{-5}^{5} \delta\,(25 - x^2)\,dx = \delta\left[25x - \frac{x^3}{3}\right]_{-5}^{5}$
$= 2\delta\left(5^3 - \frac{5^3}{3}\right) = \frac{4}{3}\delta \cdot 5^3$. Therefore $\bar{y} = \frac{M_x}{M} = \frac{\delta \cdot 5^4 \cdot \left(\frac{8}{3}\right)}{\delta \cdot 5^3 \cdot \left(\frac{4}{3}\right)} = 10$. The plate's center of mass is the point $(\bar{x}, \bar{y}) = (0, 10)$.

3. Intersection points: $x - x^2 = -x \Rightarrow 2x - x^2 = 0$ $\Rightarrow x(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{(x - x^2) + (-x)}{2}\right)$ $= \left(x, -\frac{x^2}{2}\right)$, length: $(x - x^2) - (-x) = 2x - x^2$, width: dx, area: $dA = (2x - x^2)\,dx$, mass: $dm = \delta\,dA = \delta\,(2x - x^2)\,dx$. The moment of the strip about the x-axis is



$\tilde{y}\,dm = \left(-\frac{x^2}{2}\right)\delta\,(2x - x^2)\,dx$; about the y-axis it is $\tilde{x}\,dm = x \cdot \delta\,(2x - x^2)\,dx$. Thus, $M_x = \int \tilde{y}\,dm$
$= -\int_0^2 \left(\frac{\delta}{2}x^2\right)(2x - x^2)\,dx = -\frac{\delta}{2}\int_0^2 (2x^3 - x^4)\,dx = -\frac{\delta}{2}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = -\frac{\delta}{2}\left(2^3 - \frac{2^5}{5}\right) = -\frac{\delta}{2} \cdot 2^3\left(1 - \frac{4}{5}\right)$
$= -\frac{4\delta}{5}$; $M_y = \int \tilde{x}\,dm = \int_0^2 x \cdot \delta\,(2x - x^2)\,dx = \delta\int_0^2 (2x^2 - x^3) = \delta\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \delta\left(2 \cdot \frac{2^3}{3} - \frac{2^4}{4}\right) = \frac{\delta \cdot 2^4}{12} = \frac{4\delta}{3}$;
$M = \int dm = \int_0^2 \delta\,(2x - x^2)\,dx = \delta\int_0^2 (2x - x^2)\,dx = \delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M}$
$= \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ and $\bar{y} = \frac{M_x}{M} = \left(-\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = -\frac{3}{5} \Rightarrow (\bar{x}, \bar{y}) = \left(1, -\frac{3}{5}\right)$ is the center of mass.

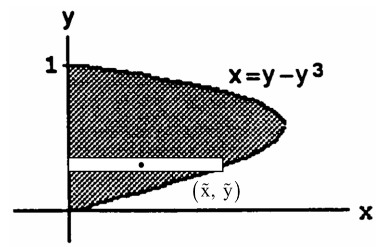
4. Intersection points: $x^2 - 3 = -2x^2 \Rightarrow 3x^2 - 3 = 0$ $\Rightarrow 3(x - 1)(x + 1) = 0 \Rightarrow x = -1$ or $x = 1$. Applying the symmetry argument analogous to the one in Exercise 1, we find $\bar{x} = 0$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{-2x^2 + (x^2 - 3)}{2}\right) = \left(x, \frac{-x^2 - 3}{2}\right)$, length: $-2x^2 - (x^2 - 3) = 3(1 - x^2)$, width: dx, area: $dA = 3(1 - x^2)\,dx$, mass: $dm = \delta\,dA = 3\delta\,(1 - x^2)\,dx$. The moment of the strip about the x-axis is



$\tilde{y}\,dm = \frac{3}{2}\delta\,(-x^2 - 3)(1 - x^2)\,dx = \frac{3}{2}\delta\,(x^4 + 3x^2 - x^2 - 3)\,dx = \frac{3}{2}\delta\,(x^4 + 2x^2 - 3)\,dx$; $M_x = \int \tilde{y}\,dm$
$= \frac{3}{2}\delta\int_{-1}^{1}(x^4 + 2x^2 - 3)\,dx = \frac{3}{2}\delta\left[\frac{x^5}{5} + \frac{2x^3}{3} - 3x\right]_{-1}^{1} = \frac{3}{2} \cdot \delta \cdot 2\left(\frac{1}{5} + \frac{2}{3} - 3\right) = 3\delta\left(\frac{3 + 10 - 45}{15}\right) = -\frac{32\delta}{5}$;

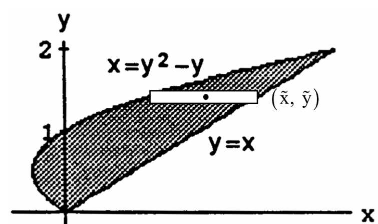
$M = \int dm = 3\delta \int_{-1}^{1} (1 - x^2)\, dx = 3\delta \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 3\delta \cdot 2\left(1 - \frac{1}{3}\right) = 4\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = -\frac{\delta \cdot 32}{5 \cdot \delta \cdot 4} = -\frac{8}{5}$ $\Rightarrow (\overline{x}, \overline{y}) = \left(0, -\frac{8}{5}\right)$ is the center of mass.

5. The typical *horizontal* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(\frac{y - y^3}{2}, y\right)$, length: $y - y^3$, width: dy, area: $dA = (y - y^3)\, dy$, mass: $dm = \delta\, dA = \delta\,(y - y^3)\, dy$. The moment of the strip about the y-axis is $\widetilde{x}\, dm = \delta \left(\frac{y - y^3}{2}\right)(y - y^3)\, dy = \frac{\delta}{2}(y - y^3)^2\, dy$ $= \frac{\delta}{2}(y^2 - 2y^4 + y^6)\, dy$; the moment about the x-axis is



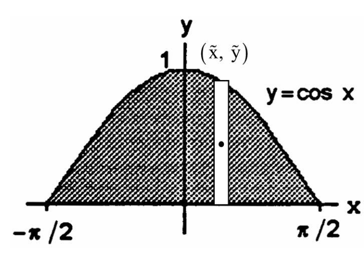
$\widetilde{y}\, dm = \delta y\,(y - y^3)\, dy = \delta\,(y^2 - y^4)\, dy$. Thus, $M_x = \int \widetilde{y}\, dm = \delta \int_0^1 (y^2 - y^4)\, dy = \delta \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \delta\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\delta}{15}$; $M_y = \int \widetilde{x}\, dm = \frac{\delta}{2} \int_0^1 (y^2 - 2y^4 + y^6)\, dy = \frac{\delta}{2}\left[\frac{y^3}{3} - \frac{2y^5}{5} + \frac{y^7}{7}\right]_0^1 = \frac{\delta}{2}\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{\delta}{2}\left(\frac{35 - 42 + 15}{3 \cdot 5 \cdot 7}\right) = \frac{4\delta}{105}$; $M = \int dm$ $= \delta \int_0^1 (y - y^3)\, dy = \delta \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \delta\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\delta}{4}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{105}\right)\left(\frac{4}{\delta}\right) = \frac{16}{105}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{2\delta}{15}\right)\left(\frac{4}{\delta}\right)$ $= \frac{8}{15} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{16}{105}, \frac{8}{15}\right)$ is the center of mass.

6. Intersection points: $y = y^2 - y \Rightarrow y^2 - 2y = 0$ $\Rightarrow y(y - 2) = 0 \Rightarrow y = 0$ or $y = 2$. The typical *horizontal* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(\frac{(y^2 - y) + y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$, length: $y - (y^2 - y) = 2y - y^2$, width: dy, area: $dA = (2y - y^2)\, dy$, mass: $dm = \delta\, dA = \delta\,(2y - y^2)\, dy$. The moment about the y-axis is $\widetilde{x}\, dm = \frac{\delta}{2} \cdot y^2 (2y - y^2)\, dy$



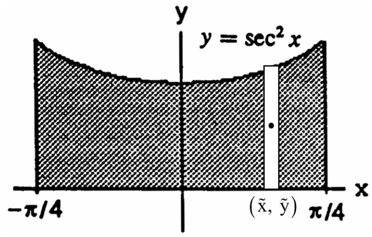
$= \frac{\delta}{2}(2y^3 - y^4)\, dy$; the moment about the x-axis is $\widetilde{y}\, dm = \delta y\,(2y - y^2)\, dy = \delta\,(2y^2 - y^3)\, dy$. Thus, $M_x = \int \widetilde{y}\, dm = \int_0^2 \delta\,(2y^2 - y^3)\, dy = \delta \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \delta\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{16\delta}{12}(4 - 3) = \frac{4\delta}{3}$; $M_y = \int \widetilde{x}\, dm$ $= \int_0^2 \frac{\delta}{2}(2y^3 - y^4)\, dy = \frac{\delta}{2}\left[\frac{y^4}{2} - \frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8 - \frac{32}{5}\right) = \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5}$; $M = \int dm = \int_0^2 \delta\,(2y - y^2)\, dy$ $= \delta \left[y^2 - \frac{y^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\overline{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\overline{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ $\Rightarrow (\overline{x}, \overline{y}) = \left(\frac{3}{5}, 1\right)$ is the center of mass.

7. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\overline{x} = 0$. The typical *vertical* strip has center of mass: $(\widetilde{x}, \widetilde{y}) = \left(x, \frac{\cos x}{2}\right)$, length: $\cos x$, width: dx, area: $dA = \cos x\, dx$, mass: $dm = \delta\, dA = \delta \cos x\, dx$. The moment of the strip about the x-axis is $\widetilde{y}\, dm = \delta \cdot \frac{\cos x}{2} \cdot \cos x\, dx$ $= \frac{\delta}{2}\cos^2 x\, dx = \frac{\delta}{2}\left(\frac{1 + \cos 2x}{2}\right) dx = \frac{\delta}{4}(1 + \cos 2x)\, dx$; thus,



$M_x = \int \widetilde{y}\, dm = \int_{-\pi/2}^{\pi/2} \frac{\delta}{4}(1 + \cos 2x)\, dx = \frac{\delta}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\delta}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2}\right)\right] = \frac{\delta\pi}{4}$; $M = \int dm = \delta \int_{-\pi/2}^{\pi/2} \cos x\, dx$ $= \delta[\sin x]_{-\pi/2}^{\pi/2} = 2\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = \frac{\delta\pi}{4 \cdot 2\delta} = \frac{\pi}{8} \Rightarrow (\overline{x}, \overline{y}) = \left(0, \frac{\pi}{8}\right)$ is the center of mass.

8. Applying the symmetry argument analogous to the one used in Exercise 1, we find $\overline{x} = 0$. The typical vertical strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sec^2 x}{2}\right)$, length: $\sec^2 x$, width: dx, area: $dA = \sec^2 x\, dx$, mass: $dm = \delta\, dA = \delta \sec^2 x\, dx$. The moment about the x-axis is $\tilde{y}\, dm = \left(\frac{\sec^2 x}{2}\right)(\delta \sec^2 x)\, dx$ $= \frac{\delta}{2} \sec^4 x\, dx$. $M_x = \int_{-\pi/4}^{\pi/4} \tilde{y}\, dm = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} \sec^4 x\, dx$

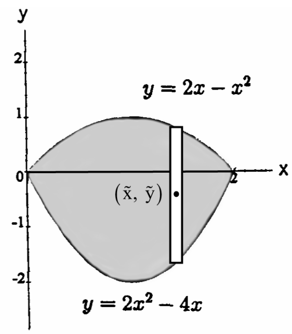


$= \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} (\tan^2 x + 1)(\sec^2 x)\, dx = \frac{\delta}{2} \int_{-\pi/4}^{\pi/4} (\tan x)^2 (\sec^2 x)\, dx + \frac{\delta}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{\delta}{2}\left[\frac{(\tan x)^3}{3}\right]_{-\pi/4}^{\pi/4} + \frac{\delta}{2}[\tan x]_{-\pi/4}^{\pi/4}$

$= \frac{\delta}{2}\left[\frac{1}{3} - \left(-\frac{1}{3}\right)\right] + \frac{\delta}{2}[1 - (-1)] = \frac{\delta}{3} + \delta = \frac{4\delta}{3}$; $M = \int dm = \delta \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \delta[\tan x]_{-\pi/4}^{\pi/4} = \delta[1 - (-1)] = 2\delta$.

Therefore, $\overline{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{1}{2\delta}\right) = \frac{2}{3} \Rightarrow (\overline{x}, \overline{y}) = \left(0, \frac{2}{3}\right)$ is the center of mass.

9. Since the plate is symmetric about the line $x = 1$ and its density is constant, the distribution of mass is symmetric about this line and the center of mass lies on it. This means that $\overline{x} = 1$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{(2x - x^2) + (2x^2 - 4x)}{2}\right) = \left(x, \frac{x^2 - 2x}{2}\right)$, length: $(2x - x^2) - (2x^2 - 4x) = -3x^2 + 6x = 3(2x - x^2)$, width: dx, area: $dA = 3(2x - x^2)\, dx$, mass: $dm = \delta\, dA$ $= 3\delta(2x - x^2)\, dx$. The moment about the x-axis is $\tilde{y}\, dm = \frac{3}{2}\delta(x^2 - 2x)(2x - x^2)\, dx = -\frac{3}{2}\delta(x^2 - 2x)^2\, dx$



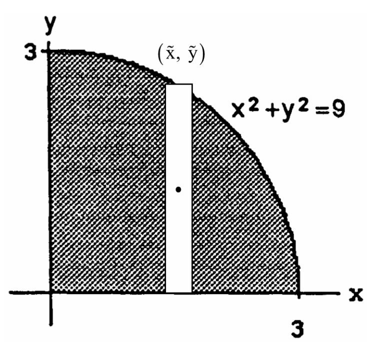
$= -\frac{3}{2}\delta(x^4 - 4x^3 + 4x^2)\, dx$. Thus, $M_x = \int \tilde{y}\, dm = -\int_0^2 \frac{3}{2}\delta(x^4 - 4x^3 + 4x^2)\, dx = -\frac{3}{2}\delta\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_0^2$

$= -\frac{3}{2}\delta\left(\frac{2^5}{5} - 2^4 + \frac{4}{3} \cdot 2^3\right) = -\frac{3}{2}\delta \cdot 2^4\left(\frac{2}{5} - 1 + \frac{2}{3}\right) = -\frac{3}{2}\delta \cdot 2^4\left(\frac{6 - 15 + 10}{15}\right) = -\frac{8\delta}{5}$; $M = \int dm$

$= \int_0^2 3\delta(2x - x^2)\, dx = 3\delta\left[x^2 - \frac{x^3}{3}\right]_0^2 = 3\delta\left(4 - \frac{8}{3}\right) = 4\delta$. Therefore, $\overline{y} = \frac{M_x}{M} = \left(-\frac{8\delta}{5}\right)\left(\frac{1}{4\delta}\right) = -\frac{2}{5}$

$\Rightarrow (\overline{x}, \overline{y}) = \left(1, -\frac{2}{5}\right)$ is the center of mass.

10. (a) Since the plate is symmetric about the line $x = y$ and its density is constant, the distribution of mass is symmetric about this line. This means that $\overline{x} = \overline{y}$. The typical *vertical* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{9 - x^2}}{2}\right)$, length: $\sqrt{9 - x^2}$, width: dx, area: $dA = \sqrt{9 - x^2}\, dx$, mass: $dm = \delta\, dA = \delta\sqrt{9 - x^2}\, dx$. The moment about the x-axis is



$\tilde{y}\, dm = \delta\left(\frac{\sqrt{9 - x^2}}{2}\right)\sqrt{9 - x^2}\, dx = \frac{\delta}{2}(9 - x^2)\, dx$. Thus, $M_x = \int \tilde{y}\, dm = \int_0^3 \frac{\delta}{2}(9 - x^2)\, dx = \frac{\delta}{2}\left[9x - \frac{x^3}{3}\right]_0^3$

$= \frac{\delta}{2}(27 - 9) = 9\delta$; $M = \int dm = \int \delta\, dA = \delta \int dA = \delta$(Area of a quarter of a circle of radius 3) $= \delta\left(\frac{9\pi}{4}\right) = \frac{9\pi\delta}{4}$.

Therefore, $\overline{y} = \frac{M_x}{M} = (9\delta)\left(\frac{4}{9\pi\delta}\right) = \frac{4}{\pi} \Rightarrow (\overline{x}, \overline{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right)$ is the center of mass.